Problem Set #8 Solutions
Part A
December 6, 2002

(1) Consider the function $f_y(x) = f(x) - y$, where $y \in \text{Image} f$ is fixed. As in the proof of the Inverse Function Theorem given in the text, we would like to see if we can solve $f_y(x) = 0$ in some neighborhood of a_0, i.e. if Newton's method with initial guess a_0 converges, for any $y \in \mathbb{R}$. Note that $Df(x) = Df_y(x)$, so that by the first condition of the theorem $Df_y(x)^{-1}$ exists and $|Df_y(X)^{-1}| \leq K$ for every $x \in B_R(a_0)$. Furthermore if $Df(x)$ is Lipschitz with constant M on $B_R(a_0)$, then so is $Df_y(x)$, and $|f_y(x) - f_y(a_0)| = |f(x) - f(a_0)| \leq \frac{1}{4MK^2}$ for all $x \in B_R(a_0)$. Choosing y such that $|f_y(a_0)| = |f(a_0) - y| \leq \frac{1}{2MK^2}$ (i.e. $y \in B_{1/2MK^2}(f(a_0))$– this can be done since f is continuous), we have $|f_y(a_0)|K^2M \leq \frac{1}{2}$, and Kantorovich's Theorem applies. Then, as before, the solution $g(y)$ to $f_y(x) = 0$ is a local inverse to f at a_0, and f is locally invertible at a_0, as desired.

(2) (Problem 2.7.12)

(a) We have $Df\binom{x}{y} = \begin{bmatrix} -2x - \sin x & 1 \\ 1 & -2y - 2\sin y \end{bmatrix}$, where $f\binom{x}{y} = \binom{y - x^2 + 8 + \cos x}{x - y^2 + 9 + 2\cos y}$. One iteration of Newton's mehtod starting at $a_0 = \binom{\pi}{\pi}$ gives

$$\begin{aligned} a_1 &= a_0 - Df(a_0)^{-1} f(a_0) \\ &= \begin{pmatrix} \pi \\ \pi \end{pmatrix} - \begin{bmatrix} -2\pi & 1 \\ 1 & -2\pi \end{bmatrix}^{-1} \begin{pmatrix} \pi - \pi^2 + 7 \\ \pi - \pi^2 + 7 \end{pmatrix} \\ &= \begin{pmatrix} \pi \\ \pi \end{pmatrix} - \frac{\pi^2 - \pi - 7}{2\pi - 1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{\pi^2 + 7}{2\pi - 1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 3.193 \\ 3.193 \end{pmatrix}. \end{aligned}$$

(b) We would like to apply Kantorovich's Theorem here. Let $h_0 = -[Df(a_0)]^{-1} f(a_0) \approx 0.0514 \binom{1}{1}$; then $|h_0| \approx 0.0728$. Since $U = \mathbb{R}^2$ in this case, we have trivially that $U_0 = B_{|h_0|}(a_1) \subset U$. Furthermore, $\frac{\pi}{2} < \pi - 0.0729 < \pi + 0.0729 < \frac{3\pi}{2}$, so that $\cos x$ and $\cos y$ are bounded above by 0 in U_0. Now, f is clearly C^2, so we may apply Proposition 2.7.10, we have that Df is Lipschitz in U_0 with constant

$$M = \left(\sum_{1 \leq i,j,k \leq 2} (c_{i,j,k})^2 \right)^{\frac{1}{2}} = \max_{x,y \in U_0} \sqrt{(-2 - \cos y)^2 + (-2 - \cos x)^2} \leq \sqrt{2 \cdot 2^2} = 2\sqrt{2}.$$

So we have

$$\left| f \begin{pmatrix} \pi \\ \pi \end{pmatrix} \right| \cdot \left| Df \begin{pmatrix} \pi \\ \pi \end{pmatrix}^{-1} \right|^2 \cdot M < (0.272 \cdot \sqrt{2}) \cdot 0.0547 \cdot 2\sqrt{2} \approx 0.0596 < \frac{1}{2}.$$

Thus the conditions for Kantorovich's Theorem are satisfied, and so Newton's Method with initial guess a_0 converges within U_0.

(c) We know by Kantorovich's Theorem that Newton's Method converges in U_0. Note that the radius of U_0 is at $|h_0| = |a_0 - a_1| \approx 0.0728 < 0.0729 = r$, so that $B_{2r}(a_0) \supset B_r(a_1) = U_0$ contains a root of the equations, and $R = 2r = 0.1458 < 1$, as desired.

Elizabeth Schemm