# Math 25a Homework 8 Part B Solutions

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**1a.** Solving for $y$ using the quadratic formula, we get $y = -1/2 \pm \sqrt{-3x - 3/4}$. We may choose the $+$ or $-$ as needed depending on the neighborhood around $y$ that we are considering. However, at $y = -1/2$, it is not possible to choose just one branch, so $y$ is not locally a function of $x$ around $y = -1/2$. So, $y$ is implicitly a function of $x$ except at $y = -1/2$.

**1b.** The derivative of the function is $[3 \; 2y + 1]$. By the implicit function theorem, we will be able to solve for $y$ in terms of $x$ as long as the matrix $[2y + 1]$ onto. It is onto if and only if $2y + 1 \neq 0$, so $y$ is implicitly a function of $x$ for $y \neq -1/2$.

**1c.** Use theorem 2.9.13. First, we find the Lipschitz constant of the derivative. $|DF(x_1, y_1) - DF(x_2, y_2)| = |[0 \; 2y_1 - 2y_2]| = 2|y_1 - y_2| \leq 2|(x_1 \; y_1) - (x_2 \; y_2)|$. This shows the Lipschitz constant $M$ is 2.

Next, we need to find $|L^{-1}|^2$, where $L$ is $\begin{bmatrix} 3 & 2y + 1 \\ 1 & 0 \end{bmatrix}$. We are using $y = \frac{\sqrt{3}-1}{2}$. Plugging this in and computing the inverse, we get $L^{-1} = \begin{bmatrix} -\sqrt{3} & \sqrt{3}/3 \\ 1 & 0 \end{bmatrix}$, and $|L^{-1}|^2 = \frac{13}{3}$.

Using the formula $M = \frac{1}{2R|L^{-1}|^2}$, we find $R = 3/52$. So, we have an implicit function for $x \in (-1/2 - 3/52, 1/2 + 3/52)$.

**1d.** Use the same procedure. The Lipschitz constant is again $M = 2$. Compute $L^{-1}$ with $y = 5/2$ to get $L^{-1} = \begin{bmatrix} -1/2 & 1/6 \\ 1 & 0 \end{bmatrix}$ and $|L^{-1}|^2 = \frac{23}{18}$. Using the same formula as before, we compute $R = 9/46$. So, we have an implicit function for $x \in (-13/4 - 9/46, -13/4 + 9/46)$.

**2.** Yes. We use the implicit function theorem. Let $F(A) = A^2$, where $A$ is a 2x2 matrix. Consider the 2x2 matrix as a vector in $\mathbb{R}^4$. Then, by multiplying out $A^2$, we find $F(a, b, c, d) = (a^2 + bc, ab + bd, ac + cd, bc + d^2)$. Computing

the derivative, we get

$$DF(A) = \begin{bmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{bmatrix}$$

Plug in the identity matrix, $(a, b, c, d) = (1, 0, 0, 1)$, and the derivative comes out to be twice the identity, which is clearly invertible. That means we can use the inverse function theorem. In a neighborhood around $I$, the inverse of $F$ exists. Since the inverse of $F$ is the square root function, there is always a square root of $I + \epsilon B$ (for $\epsilon$ small enough to be in the neighborhood) near $I$.

**3.** Yes. Use the implicit function theorem. Define $F(x, y, a) = (x + y + \sin(xy) - a, \sin(x^2 + y) - 2a)$. The solution set to the given equations is $F(x, y, a) = (0, 0)$. The derivative of $F$ is

$$\begin{bmatrix} 1 + y\cos(xy) & 1 + x\cos(xy) & -1 \\ 2x\cos(x^2 + y) & \cos(x^2 + y) & -2 \end{bmatrix}$$

In the implicit function theorem, to determine if $x$ and $y$ are expressible as a function of $a$, we look at the matrix

$$\begin{bmatrix} 1 + y\cos(xy) & 1 + x\cos(xy) \\ 2x\cos(x^2 + y) & \cos(x^2 + y) \end{bmatrix}$$

We are looking for a solution around $a = 0$. When $a = 0$, a possible solution is $x = y = 0$. Letting $x = y = 0$, the determinant of the matrix is 1, so it is must be onto. That means the implicit function theorem applies, and there are solutions for $x$ and $y$ for $a$ in a neighborhood around $a = 0$.

**4.** We will use the inverse function theorem. We want to compute the derivative of $F(A) = A^2 + A^{-1}$. Consider the 2x2 matrices as vectors in $\mathbb{R}^4$. For convenience, set $x = ad - bc$. Then, by expanding the expression for $F(A)$, we compute $F(a, b, c, d) = (a^2 + bc + d/x, ab + bd - b/x, ac + cd - c/x, d^2 + bc + a/x)$. The derivative of this is

$$\begin{bmatrix} 2a - d^2/x^2 & c + cd/x^2 & b + bd/x^2 & -bc/x^2 \\ b + bd/x^2 & a + d - ad/x^2 & -b^2/x^2 & b + ab/x^2 \\ c + cd/x^2 & -c^2/x^2 & a + d - ad/x^2 & c + ac/x^2 \\ -bc/x^2 & c + ac/x^2 & b + ab/x^2 & 2d - a^2/x^2 \end{bmatrix}$$

Note that $x$ is non-zero for $A$ near $A_0$; i.e. $(a, b, c, d)$ near $(0, 1, -1, 0)$. Thus, the derivative is continuous. Next, we want to verify that the derivative is invertible around $A = A_0$. Plugging in values, we find

$$DF(A_0) = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

This has determinant 5, so it is invertible. All hypotheses of the inverse function theorem are satisfied, so there is a differentiable inverse map in some neighborhood of $B_0 = F(A_0)$.