Math 25a Homework 9 Part A Solutions

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1a. It is a smooth manifold except when $c = 0$. Let $F(x, y) = x^2 + y^3 - c$. We wish to show that $F(x, y) = 0$ is a smooth manifold except when $c = 0$. Then $DF(x, y) = [2x 3y^2]$. Since DF is a linear map to $\mathbb{R}$, it is onto as long as it is not always zero. It is always zero if and only if $x = y = 0$. The point $(0, 0)$ satisfies $F(x, y) = 0$ if and only if $c = 0$. Therefore, if $c \neq 0$, it is a smooth manifold by theorem 3.1.10.

It is not a smooth manifold when $c = 0$. This is because the graph has a cusp at the origin. In particular, solving for y in terms of x gives a function that is not differentiable at the origin, while solving for x in terms of y is not possible around the origin because the graph has two branches. This contradicts the definition of a manifold.

2. The tangent space at (u, v) is the kernel of $DF(u, v)$. Solving $DF(u, v) \cdot (x, y) = 0$ gives $2ux + 3v^2y = 0$. The tangent line is the tangent space translated to pass through (u, v) (rather than the origin). The equation for this line is $2u(x - u) + 3v^2(y - v) = 0$.

Note: Several people wrote the lines in the form $y = ax + b$. This doesn't always work because some of the tangent lines are vertical (namely, whenever $v = 0$).

Note 2: The tangent line does not exist for $c = 0$, $u = v = 0$ because our equation becomes $0 = 0$, which is not a line.

3a. Let the columns of A be $c_1, c_2, \ldots, c_n$. Then the rows of $A^\top$ are $c_1, c_2, \ldots, c_n$. By matrix multiplication, the ijth entry of $A^\top A$ is $c_i \cdot c_j$. The set of vectors $c_1, c_2, \ldots, c_n$ is orthonormal if and only if $c_i \cdot c_j = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Note that the ijth entry of the identity matrix is also δ_{ij}. Putting this together, we see that the columns of A are orthonormal (and thus an orthonormal basis) if and only if $A^\top A = I$.

3b. Because we have $A^\top A = I$, it must be that $A^{-1} = A^\top$. Then $(A^{-1})^\top A^{-1} = (A^\top)^\top A^\top = AA^\top = AA^{-1} = I$, which shows $A^{-1} \in O(n)$.

For the next part, we have $(AB)^\top(AB) = B^\top A^\top AB$. Now, use the fact that $A^\top A = I$ and $B^\top B = I$. $B^\top A^\top AB = B^\top B = I$, and that shows

$AB \in O(n)$.

3c. To show a matrix M is symmetric, is suffices to $M = M^\top$. We have $(A^\top A - I)^\top = (A^\top A)^\top - I^\top = A^\top A - I$, as desired.

3d. First, we claim $DF(A)H = A^\top H + H^\top A$. Note that $DF(A)$ is linear because $DF(A)xB + yC = A^\top(xB+yC)+(xB+yC)^\top A = xA^\top B + xB^\top A + yA^\top C + yC^\top A = xDF(A)B + yDF(A)C$. To show that this is the derivative, we plug it into the definition of the derivative:

$$\lim_{|H|\to 0} 1/|H|(F(A+H) - F(A) - DF(A)H) =$$

$$\lim_{|H|\to 0} 1/|H|((A+H)^\top(A+H) - I - A^\top A + I - A^\top H - H^\top A)$$

Expanding the expression, one is left with $1/|H|(H^\top H)$. The absolute value of this is $\frac{|H|^2}{|H|}$, and this goes to 0, as desired. That proves that $DF(A)H = A^\top H + H^\top A$.

Armed with this information, we can now determine when $DF(A)$ is onto. We wish to show that given any symmetric matrix M, we can find H such that $DF(A)H = M$. Set $H = (MA^{-1})^\top/2$ (we're using the fact that A is invertible). Plugging this in to our equation for $DF(A)H$ and simplifying, we get $M^\top/2 + M/2$. Since M is symmetric, this evaluates to M, and we have shown it is onto.

Note: The inspiration for the choice $H = (MA^{-1})^\top/2$ comes from the fact that $A^\top H + H^\top A = A^\top H + (A^\top H)^\top$. If we have $A^\top H = M/2$, then it's easy to see this expression will evaluate to M.

3e. $O(n)$ is the set $F(X) = 0$. Note that all elements of $O(n)$ are invertible, since an orthogonal matrix has linearly independent columns. Then, using part (d) and theorem 3.1.10, we find that $O(n)$ is a manifold.

For the next part, we have $T_I O(n) = \ker DF(A)I$. Using our expression for $DF(A)H$, we see $DF(A)I = A + A^\top$. This is zero if and only if $A = -A^\top$, i.e. if A is anti-symmetric.