Problem Set #9 Solutions
Part B
December 18, 2002

(1) (Problem 3.1.8)

(a) Consider the equations $F_a(x,y,z) = x - y^2 - a = 0$ and $F_b(x,y,z) = x^2 + y^2 + z^2 - b = 0$ in $\mathbb{R}^3$. First note that F_b has no solutions for $b < 0$, so we will consider only the cases where $b \geq 0$ Then $DF_a = \begin{bmatrix} 1 & -2y & 0 \end{bmatrix}$ is always onto, and $DF_b = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$ is onto whenever $(x,y,z) \neq (0,0,0)$. Therefore, by Theorem 3.1.10 X_a given by the points where $F_a = 0$ is always a smotth surface in $\mathbb{R}^3$, and $_b$ is a smooth surface whenever $b > 0$.

(b) Now consider the function

$$F_{a,b}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y^2 - a \\ x^2 + y^2 + z^2 - b \end{pmatrix}$$

as a map from $\mathbb{R}^3$ to $\mathbb{R}^2$. $X_a \cap Y_b$ is then the set of solutions to the equation $F_{a,b}(x,y,z) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. As before, we must have $b \geq 0$. By Theorem 3.1.10, $X_a \cap Y_b$ is a smooth surface whenever

$$DF_{a,b}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2y & 0 \\ 2x & 2y & 2x \end{pmatrix}$$

is onto, which occurs whenever any of the 2×2 submatrices of $DF_{a,b}$ is invertible. Thus our only difficulty lies in the case where $2y + 4xy = 2z = -4yz = 0$. There are two possibilities here: either $z = 0$ and $y = 0$, or $z = 0$ and $x = -\frac{1}{2}$.

Now, if $y = z = 0$ we have from our original equations that $x - a = 0$ and $x^2 - b = 0$, so that DF is not invertible when $a^2 = b$. Furthermore, a is the smallest value that x takes on X_a, and $\sqrt{b}$ is the greatest value that x takes on Y_b. So we also have that for $b \leq a^2$, $X_a \cap Y_b = \emptyset$, and hence does not give a smooth curve in $\mathbb{R}^3$.

If $x = -\frac{1}{2}$ and $z = 0$ we have $y^2 = -a - \frac{1}{2}$ and $y^2 = b - \frac{1}{4}$, or DF is not onto when $b = -a - \frac{1}{4}$. Also, when $b < -a - \frac{1}{4}$, there are no solutions to $F = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, so $X_a \cap Y_b$ is a nontrivial smooth surface whenever $b > \max\{-a - \frac{1}{4}, a^2, 0\}$.

(c) Geometrically speaking, observe that by taking projections of the locus of points $F_{a,b} = 0$ in the plane $z = 0$, $X_a \cap Y_b$ is not guaranteed to be a smooth curve when X_a is either disjoint from or tangent (at at least one point) to Y_b.

(2) (Problem 3.1.16)

(a) X_l is given by the zero locus of the function $F : \mathbb{R}^4 \to \mathbb{R}^2$, where

$$F\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (x-t)^2 + y^2 + z^2 - l^2 \\ x^2 + y^2 + z^2 - 1 \end{pmatrix}$$

is a C^1 function.

(b) Here we have

$$DF = \begin{bmatrix} -2(x-t) & 2(x-t) & 2y & 2z \\ 0 & 2x & 2y & 2z \end{bmatrix}.$$

At the point $p = \begin{pmatrix} 1+l \\ 1 \\ 0 \\ 0 \end{pmatrix}$, we have

$$DF(p) = \begin{bmatrix} 2l & -2l & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

which is onto $\mathbb{R}^2$ because it has two pivotal columns (we assume, of course, that $l > 0$). Therefore X_l is locally a manifold at the point p.

(c) In general, DF is onto so long as not all of the 2×2 submatrices of DF are not invertible. To find where DF is not onto, we consider the case where $x(t-x) = y(t-x) = z(t-x) = yt = zt = 0$. If $t - x \neq 0$, then we must have $x = y = z = 0$; however, this does not lie on X_l. Therefore $t - x = 0$. If $t \neq 0$, we must have $y = z = 0$, and again there are no points on X_l satisfying such constraints. Finally, if $t = 0$ the equations for X_l give $l^2 = 1$; assuming that l is positive, then, we have $l = 1$ is the only case where X_l is not a manifold.

(d) Let $l = 1$. Then for any points on X_1 where $x = t = 0$, we have

$$DF = \begin{bmatrix} 0 & 0 & 2y & 2z \\ 0 & 0 & 2y & 2z \end{bmatrix}$$

which is clearly not onto for any choice of yy and z. Thus the set $\left\{ \begin{pmatrix} 0 \\ 0 \\ y \\ z \end{pmatrix} \right\}$ gives one family of points near which X_1 is not a manifold; there are of course several others.

(3) (Problem 3.2.8) Recall that $T_pX_l = \ker[DF(p)]$, so that by part (b) of the previous problem we see that

$$T_pX_l = \left\{ \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \middle| \begin{array}{c} 2lt - 2lx = 0 \\ \text{and} \\ 2x = 0 \end{array} \right\}$$

In other words, $x = t = 0$, and y and z are unconstrained. This defines T_pX_l.

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