Math 25a Homework 2 Part B Solutions

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1. For all parts, $d(p,q) \geq 0$ is obvious. Also, because $(p - q)^2 = (q - p)^2$ and $|p - q| = |q - p|$, $d(p, q) = d(q, p)$ in all parts. So, we have two properties to prove for each part. (Note: we will refer to the fourth property as the triangle inequality.)

1a. $d_2(p, q) = 0$ if and only if $(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2 = 0$. Since each term is nonnegative, all terms must be 0. That is true if and only if $p_i = q_i$ for all i, i.e. $p = q$.

Observe that $d_2(p, q) = |p - q|$, as we defined distance in class. By the triangle inequality for vectors, $|p - q| \leq |p - r| + |r - q|$, which implies $d_2(p, q) \leq d_2(p, r) + d_2(r, q)$.

1b. Since it is the sum of nonnegative terms, $d_1(p, q) = 0$ if and only if all terms are 0. That is true if and only if $p_i = q_i$ for all i. Equivalently, $p = q$.

To prove the triangle inequality, we have $d_1(p, q) = |p_1 - q_1| + \cdots + |p_n - q_n| \leq |p_1 - r_1| + |r_1 - q_1| + \cdots + |p_n - r_n| + |r_n - q_n| = d_1(p, r) + d_1(r, q)$.

1c and 1e. The maximum value of a set of nonnegative numbers is 0 if and only if all all elements are 0. Equivalently, $p_i = q_i$ for all i (in part 1c) or $f(x) = g(x)$ for all x (in part 1e). In other words, $p = q$ or $f = g$, respectively.

To get the triangle inequality, we only need to show $\max |x_i + y_i| \leq \max |x_i| + \max |y_i|$, where i ranges over any set. This must be true because $\max |x_i + y_i| \leq \max(|x_i| + |y_i|) = |x_k| + |y_k|$ for some k, and $|x_k| \leq \max |x_i|$ and $|y_k| \leq \max |y_i|$. Then, we obtain the triangle inequality for 1c by plugging in $x_i = p_i - r_i$ and $y_i = r_i - q_i$ for $1 \leq i \leq n$, and for 1e by plugging in $x_i = f(i) - h(i)$ and $y_i = h(i) - g(i)$ where i ranges over $[0, 1]$.

Remark: $\max_x |f(x) - g(x)|$ must exist because $|f(x) - g(x)|$ is a continuous function on a compact interval, so it attains a maximum value (this was shown in class). If the functions weren't defined on a compact set, then the metric d_∞ wouldn't make sense because sometimes its value wouldn't exist. For example, if $|f(x) - g(x)| = |x|$, this attains no maximum for unbounded sets in $\mathbb{R}$.

1d. The triangle inequality follows immediately from the fact that $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$ for all x.

If $f = g$, it is easy to see $d_1(f, g) = 0$.

If $f \neq g$, let $h(x) = |f(x) - g(x)|$. $h(x)$ is also continuous. There must be some a such that $h(a) > 0$. We wish to show $\int_0^1 h(x)dx > 0$. By the definition of continuity at a, there is some interval (b, c) such that $|h(x) - h(a)| < h(a)/2$ for $x \in (b, c)$, where we have chosen $\epsilon = h(a)/2$. Then, $h(x) > h(a)/2$ on this interval. Therefore, $\int_b^c h(x)dx > (c - b)h(a)/2 > 0$, which implies $\int_0^1 h(x)dx > 0$. Therefore, $d_1(f, g) = 0$ if and only if $f = g$.

2. We have $d_T(p, q) = d(p, q)$ for all $p, q \in T$. Since all the properties of a metric space are satisfied for all $p, q \in S$ with the function d, they are also satisfied for all $p, q \in T$ with the function d. Since $d = d_T$ on this domain, d_T satisfies the necessary properties for all $p, q \in T$.

3. (graphs of $|x| + |y| < 1$, $x^2 + y^2 < 1$, and $\max\{|x|, |y|\} < 1$)

4a. Assume the sequence $\{a_i\}$ converges to a under the metric d_1. Given some $\epsilon > 0$, consider $B_\epsilon^2(a)$. By the equivalence of d_1 and d_2, there exists a ball $B_{\epsilon_2}^1(a)$ contained in $B_\epsilon^2(a)$. By the convergence of $\{a_i\}$, there is some N such that $a_i \in B_{\epsilon_2}^1(a)$ whenever $i > N$. Thus, whenever $i > N$, $a_i \in B_\epsilon^2(a)$. That implies $\{a_i\}$ converges to a under the metric d_2.

Repeating the previous argument but interchanging d_1 and d_2 completes the proof.

4b. Without loss of generality, we can take all balls to be centered at the origin. We may do this because $d(p - a, q - a) = d(p, q)$ for each of the metrics we are considering, so if $B_r^i(0) \subset B_R^j(0)$, then $B_r^i(a) \subset B_R^j(a)$. For brevity, we will use B_r^i to denote $B_r^i(0)$.

We will prove that $B_r^1 \subset B_r^2 \subset B_r^\infty \subset B_{nr}^1$. Using this chain of inclusions, we can find a ball in one metric that is inside a given ball of any other metric. Thus, proving this will prove the equivalence of the metrics.

$B_r^1 \subset B_r^2$: By expansion, it is easy to see that $\sum |x_i|^2 < (\sum |x_i|)^2$ for any finite set of numbers x_i. Thus, $p \in B_r^1 \Rightarrow \sum_{i=1}^n |p_i| < r \Rightarrow (\sum_{i=1}^n |p_i|)^2 < r^2 \Rightarrow \sum_{i=1}^n |p_i|^2 < r^2 \Rightarrow \sqrt{p_1^2 + \cdots + p_n^2} < r \Rightarrow p \in B_r^2$.

$B_r^2 \subset B_r^\infty$: Because $|x| \leq \sqrt{x^2 + a}$ for any nonnegative a, we have $\max x_i \leq \sqrt{\sum x_i^2}$ for any finite set of numbers x_i. Thus, $p \in B_r^2 \Rightarrow \sqrt{p_1^2 + \cdots + p_n^2} < r \Rightarrow \max\{|p_1|, \ldots, |p_n|\} < r \Rightarrow p \in B_r^\infty$.

$B_r^\infty \subset B_{nr}^1$: Plainly, $\sum_{i=1}^n |p_i| \leq n \max\{|p_1|, \ldots, |p_n|\}$. Thus, $p \in B_r^\infty \Rightarrow n \max\{|p_1|, \ldots, |p_n|\} < nr \Rightarrow \sum_{i=1}^n |p_i| < nr \Rightarrow p \in B_{nr}^1$.

That completes the proof.

4c. Consider the sequence of functions defined for $i \in \mathbb{N}$ by $f_i(x) = i - i^3 x$

for $0 \leq x \leq \frac{1}{i^2}$ and $f_i(x) = 0$ for $\frac{1}{i^2} \leq x \leq 1$. It is easy to check that this function is continuous.

We claim that the limit of this sequence is $g(x) = 0$ under the first metric. We calculate $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx = \int_0^1 f(x) dx = \int_0^{1/i^2} i - i^3 x \, dx = \frac{1}{2i}$. As $i \to \infty$, this approaches 0, so the sequence converges to g.

However, under the second metric, the sequence does not converge. We show this by demonstrating that it is not Cauchy: $d_\infty(f_i, f_j) = \max_x |f_i(x) - f_j(x)| \geq |f_i(0) - f_j(0)| = |i - j|$. So given any N, we can find $i, j > N$ such that $d_\infty(f_i, f_j)$ is arbitrarily large, thus contradicting the definition of a Cauchy sequence.

5. If a sequence is Cauchy in d_2, then given any $\epsilon > 0$, we have $d_2(p_i, p_j) < \epsilon$ for $i, j > N$. Since $d_\infty(p_i, p_j) \leq d_2(p_i, p_j)$ (since $\max |p_i| \leq \sqrt{p_1^2 + \cdots + p_n^2}$), we have $d_\infty(p_i, p_j) < \epsilon$ for $i, j > N$. Therefore, if a sequence is Cauchy in d_2, it is Cauchy in d_∞. Problem 4 tells us that a sequence converges in d_2 if and only if it converges in d_∞. Hence, $(\mathbb{R}^n, d_\infty)$ is complete only if $(\mathbb{R}^n, d_2)$ is.

Let $\{a_i\}$ be a Cauchy sequence under d_2, so it is also Cauchy under d_∞. Let a_i be $(a_{i1}, a_{i2}, \ldots, a_{in})$, a vector with n components. We claim that the sequences $\{a_{ik}\}_i$ are Cauchy for every $k = 1, 2, \ldots, n$ (less formally, each component of the sequence $\{a_i\}$ is a Cauchy sequence of real numbers). By definition of Cauchy under d_∞, given $\epsilon > 0$, there is an N such that $d_\infty(a_i, a_j) < \epsilon$ for all $i, j > N$. Because $d_\infty(a_i, a_j) \geq |a_{ik} - a_{jk}|$ for each k, we have $|a_{ik} - a_{jk}| < \epsilon$ for all $i, j > N$. Therefore, the sequences $\{a_{ik}\}_i$ are Cauchy.

Let b_k be the limit of $\{a_{ik}\}_i$, which exists since $\mathbb{R}$ is complete. We claim the vector $b = (b_1, b_2, \ldots, b_n)$ is the limit of $\{a_i\}$. By definition of convergence of a sequence of real numbers, given $\epsilon > 0$, for each k there exists N_k such that $|a_{ik} - b_k| < \epsilon$ for all $i > N_k$. Let N be the greatest of all the N_k (which exists since there are only finitely many N_k). Then, $|a_{ik} - b_k| < \epsilon$ for all $i > N$ so $\max_k |a_{ik} - b_k| < \epsilon$ for all $i > N$. Notice that $\max_k |a_{ik} - b_k| = d_\infty(a_i, b)$. Therefore, we have proven that for all $\epsilon > 0$, there is an N such that $d_\infty(a_i, b) < \epsilon$ for all $i > N$. That means b is the limit of $\{a_i\}$, so $(\mathbb{R}^n, d_\infty)$ is complete. Conclude that $(\mathbb{R}^n, d_2)$ is complete, as desired.

6a. First, note that the problem should read $d(f(p), f(q)) \leq Cd(p, q)$, because strict inequality could not hold when $p = q$.

Assume there are two fixed points x and y. Then $d(f(x), f(y)) = d(x, y) > Cd(x, y)$, a contradiction.

6b. Note that for $n \in \mathbb{N}$, $d(a_{i+2}, a_{i+1}) = d(f(a_{i+1}), f(a_i)) \leq Cd(a_{i+1}, a_i)$. An easy induction proves that $d(a_{i+2}, a_{i+1}) \leq C^i d(a_2, a_1)$.

Next, we find the distance between $d(a_j, a_i)$, $j > i$. By the triangle inequality, $d(a_j, a_i) \leq d(a_i, a_{i+1}) + d(a_{i+1}, a_{i+2}) + \cdots + d(a_{j-1}, a_j)$ Applying our previous inequality, the right side is at most $(C^{i-1} + C^i + \cdots + C^{j-2})d(a_2, a_1)$.

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Clearly, this is less than $(C^{i-1} + C^i + \cdots)d(a_2, a_1) = \frac{C^{i-1}}{1-C}d(a_2, a_1)$.

$C < 1$ so letting i become arbitrarily large, C^{i-1} becomes arbitrarily small. Since $1 - C$ and $d(a_2, a_1)$ are constants, as $i \to \infty$, $\frac{C^{i-1}}{1-C}d(a_2, a_1) \to 0$.

Finally, we prove this sequence is Cauchy. Given $\epsilon > 0$, we can find an $N > 1$ such that $\frac{C^{N-1}}{1-C}d(a_2, a_1) < \epsilon$. Then, by our preceding argument, $d(a_j, a_i) \leq \frac{C^{i-1}}{1-C}d(a_2, a_1) < \epsilon$ for all $j > i > N$. This proves that $\{a_i\}$ is a Cauchy sequence.

6c. Let a be the limit of the sequence in the previous problem, which exists since it was given that the space is complete. For any $i > 1$, we have $d(a, f(a)) \leq d(a, a_i) + d(a_i, f(a)) = d(a, a_i) + d(f(a_{i-1}), f(a)) \leq d(a, a_i) + Cd(a_{i-1}, a)$.

We wish to show that we can make $d(a, f(a))$ smaller than any given $\epsilon > 0$. Since a is the limit of $\{a_i\}$, there is an N such that $d(a, a_i) < \epsilon/2$ whenever $i > N$. Then, $d(a, a_i) < \epsilon/2$ and $d(a, a_{i-1}) < \epsilon/2$ for large enough i. Adding, we obtain $d(a, a_i) + d(a, a_{i-1}) < \epsilon$. Finally, we have $d(a, f(a)) \leq d(a, a_i) + Cd(a_{i-1}, a) < d(a, a_i) + d(a, a_{i-1}) < \epsilon$.

That proves $d(a, f(a))$ is smaller than any positive number. Therefore, the quantity must be 0, and $a = f(a)$.