Review Sheet for Midterm 1

October 12, 2002

Here is a short guide to the topics we have covered so far in Math 25, along with some practice problems and solutions. Note that this is only a guide, and *not* an exhaustive list of what you need to know for the midterm. When possible we have included page references in Hubbard and Hubbard to supplement your class notes. For this exam you will be responsible for the material covered in class up to (but not including) the discussion of differential calculus and partial derivatives begun on Oct. 9. Roughly speaking, the corresponding sections in the text are 0.3-0.5, 1.0-1.3, and 1.5-1.6, although some topics we have discussed are not in these. You should also be comfortable with the basic methods of writing proofs and with the ideas introduced in the first three problem sets, although the exam questions should be somewhat easier than the problems on the homework. While you review be sure that you not only know what each of these definitions or theorems says, but also how to apply them. And as always, feel free to ask us if you have any questions.

Good Luck!

–Luke Gustafson and Elizabeth Schemm

Terms, Concepts, and Theorems

1. Objects in mathematics. You should know what the following constructions are:
   (a) Sets
   (b) Fields (namely $\mathbb{R}$ and $\mathbb{Q}$)
   (c) Sequences, subsequences
   (d) Vectors, vector spaces, and vector subspaces (p. 33-41, 225)
   (e) Metric spaces

2. Definitions. Know the definition and properties of:
   (a) Injective, surjective, and bijective maps. (p. 14)
   (b) Equivalence relations and equivalence classes.
   (c) Linear maps (linear transformations). Image and kernel of a linear map. (pp.206-207, 226)
   (d) Upper bound, least upper bound. (pp. 18-19)
   (e) Monotone increasing/decreasing functions and sequences.
   (f) Convergent/Cauchy sequences ($\varepsilon - N$ definition). (pp. 19, 94)
   (g) Norms and metrics in $\mathbb{R}^n$.
   (h) Continuity and uniform continuity ($\varepsilon - \delta$ definition). (pp. 102-104)
   (i) Cartesian product of vector spaces.
   (j) Matrices over $\mathbb{R}$ (including definition of addition and multiplication using (a_{ij}) notation). Transpose and inverse matrices (but not how to find the inverse of a $n \times n$ matrix). (pp. 43-52)
   (k) Open, closed, bounded, and compact sets. (pp. 90-91, 111-114) (also good to know are the notions of closure, interior, and boundary on pp. 92-93)

3. Some theorems
   (a) Let $\{a_n\}$ be a nondecreasing sequence that is bounded from above. Then $\{a_m\}$ has a limit. (p. 20)
   (b) Intermediate Value Theorem (p. 21)
   (c) Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by a $n \times m$ matrix, and every $n \times m$ matrix with real entries defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. (pp. 59-62)
   Warning: It is often easier and better to prove statements about linear transformations on real vector spaces using only the definition of a linear transformation, rather than relying on matrices.
   (d) The (Cauchy-) Schwarz inequality: $|v \cdot w| \leq |v|\,|w|$, for all $v, w \in \mathbb{R}^n$. (p. 75)
   (e) (Heine-Borel) A subset $C \subset \mathbb{R}^n$ is compact (i.e. every open cover of C has a finite subcover) if and only if it is closed and bounded.
   (f) (Bolzano-Weierstrass) A subset $C \subset \mathbb{R}^n$ is compact if and only if it is sequentially compact (i.e. every sequence in C has a subsequence converging to some element of C). (p. 111)

Practice Problems

1. Let $f : S \to T$ be a function. Prove that
   (a) $f^{-1}(f(A)) = A$ for all $A \subset S$ if and only if f is injective.
   (b) $f(f^{-1}(A)) = A$ for all $A \subset T$ if and only if f is surjective.
2. Prove that $\mathbb{Z}$ is complete (under the usual metric $d(a, b) = |a - b|$).
3. Is the set $\{n \in \mathbb{R} \mid n \in \mathbb{N}\}$ is closed? Is the set $\{1/n \mid n \in \mathbb{N}\}$ closed?
4. Let V, W be vector spaces and $T : V \to W$ be a linear map. Recall that $\ker T = T^{-1}(0)$. Define an equivalence relation on V so that $v \sim w$ iff $v - w \in \ker T$ (from results of the second homework, this is an equivalence relation).
   (a) Show that $v \sim w$ if and only if $T(v) = T(w)$.
   (b) Recall problem 8 from Part A of the first homework assignment. It gives us an injective map $\overline{T} : V/\sim \to W$. Prove that this is a linear map.
5. It was shown in class that if $C_0 \supset C_1 \supset \cdots$ are nonempty compact sets, then their intersection is nonempty. Find a conterexample to this result if the sets are not compact: if $C_0 = \mathbb{R}$ and C_i are all nonempty (and not required to be compact) for $i \in \mathbb{N}$.
6. It was shown in class that for a continous fuction $f : \mathbb{R}^n \to \mathbb{R}^m$ and any compact set $C \subset \mathbb{R}^n$, $f(C)$ is also compact. Consider the following related statements, and show that they are false:
   (a) For all compact sets $C \subset \mathbb{R}^m$, $f^{-1}(C)$ is compact.
   (b) For all closed sets $C \subset \mathbb{R}^n$, $f(C)$ is closed.

Solutions

1. (a) Consider $t \in f^{-1}(f(A))$. Then, $f(t) \in f(A)$. That means for some $s \in A$, $f(t) = f(s)$. By injectivity, $s = t$, so $t = s \in A$. Next, consider $t \in A$. Then, $f(t) \in f(A)$, so $t \in f^{-1}(f(A))$. Thus, if f is injective, then $f^{-1}(f(A)) = A$.
   If f is not injective, then let $f(a) = f(b)$ for $a \neq b$. Let $A = \{a\}$. Then $f^{-1}(f(a))$ contains a and b, so $f^{-1}(f(A)) \neq A$.
   (b) Consider $t \in f(f^{-1}(A))$. Then $t = f(a)$ for some $a \in f^{-1}(A)$. That means $f(a) \in A$, so $t \in A$. Next, consider $t \in A$. If f is surjective, then $t = f(s)$ for some $s \in S$. So, $s \in f^{-1}(A)$, and $t = f(s) \in f(f^{-1}(A))$.
   If f is not surjective, then let $a \in S$ be an element not in the image of f. Let $A = \{a\}$. Then, $f^{-1}(A) = \emptyset$, so $f(f^{-1}(A))$ is empty, so $f(f^{-1}(A)) \neq A$.
2. Let $\{a_i\}$ be a Cauchy sequence of integers. Then, there exists an N such that $|a_i - a_j| < 1/2$ for all $i, j > N$. Since a_i, a_j are integers, this means $a_i = a_j$ for all $i, j > N$. Therefore, $\{a_i\}$ is equal to a constant a for $i > N$, and so it converges to a. All Cauchy sequences converge, so $\mathbb{Z}$ is complete.
3. Recall that a subset of $\mathbb{R}^n$ is closed if and only if it contains all its limit points. Since the set $\{n \in \mathbb{R} \mid n \in \mathbb{N}\}$ contains all its limit points by the above argument, it is closed. Since the sequence $\{1/i\}_i$ converges to 0, and $0 \notin \{1/n \mid n \in \mathbb{N}\}$, $\{1/n \mid n \in \mathbb{N}\}$ cannot be closed. As it turns out, the set $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ is closed because it does contain all its limit points.
4. (a) $v - w \in \ker T$ if and only if $T(v - w) = 0$. Since T is linear, this is equivalent to $T(v) - T(w) = 0$, which is $T(v) = T(w)$.
   (b) By problem 5 of part A of the second homework, you know that $V/\sim$ is a vector space, with addition and multiplication defined using representatives. This makes it easy to verify that $\overline{T}$ is a linear map. Let $\overline{x}$ denote the equivalence class in $S/\sim$ given by x. For all $u, v \in V$, we have $\overline{T}(\overline{u}+\overline{v}) = \overline{T}(\overline{u+v}) = T(u+v) = T(u)+T(v) = \overline{T}(\overline{u})+\overline{T}(\overline{v})$. For all $\lambda \in \mathbb{R}$ and $v \in V$, we have $\overline{T}(\lambda\overline{v}) = \overline{T}(\overline{\lambda v}) = T(\lambda v) = \lambda T(v) = \lambda\overline{T}(\overline{v})$. Therefore, $\overline{T}$ is a linear map.
5. Take $C_i = \{x \in \mathbb{R} \mid |x| \geq i\}$. Obviously, the C_i are all nonempty. Their intersection is empty, however, because for any $x \in \mathbb{R}$, x is not in C_i for $i > |x|$.
6. (a) Take $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$, and $C = [0, 1]$. Then $f^{-1}(C) = \mathbb{R}$ is not compact.
   (b) Take $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1/(x^2 + 1)$. Take $C = \mathbb{R}$. Then, $f(C) = (0, 1]$, which is not closed.