EXTRA PROBLEMS

MATH 134
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1. Let M and N be C^∞ manifolds of dimensions m and n respectively. Prove that $M \times N$ is a C^∞ manifold of dimension $m+n$ with smooth structure determined by coordinate neighborhoods of the form $\{(U \times V, \phi \times \psi)\}$, where (U, ϕ) and (V, ψ) are coordinate neighborhoods on M and N respectively and $(\phi \times \psi)(p, q) = (\phi(p), \psi(q))$ in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.

2. Look at the "figure eight" image $\tilde{N}$ of $\mathbb{R}$ found in Example 4.9 of the single sheet handout from Boothby's book. Let the topology and the smooth structure be given by the one to one immersion G. Show that reflection $H : (X^1, x^2) \to (x^1, -x^2)$ of $\mathbb{R}^2$ in the x^1 axis, although it maps $\tilde{N}$ onto $\tilde{N}$, is not a diffeomorphism of $\tilde{N}$.

3. Let $F_i : N_i \to M$ $(i = 1, 2)$ be one-to-one immersions. Define them to be equivalent if there is a diffeomorphism $G : N_1 \to N_2$ such that $F_1 = F_2 \circ G$. Show by example that there exist inequivalent one to one immersions of $\mathbb{R}$ into $\mathbb{R}^2$ with the same image.

4. Show that if M is a connected orientable manifold then it has exactly two orientations.

5. Let M_1 and M_2 be two smooth manifolds and let $F : M_1 \to M_2$ be a diffeomorphism. Verify that M_1 is orientable if and only if M_2 is orientable.

6. The tangent bundle is defined as follows:

$$T(M) = \{X_p \in T_p(M) | \, p \in M\} = \bigcup_{p \in M} T_p(M).$$

Given an atlas $\{(U, \phi)\}$ on M we see that vectors X_p, $p \in U$ are in one-to-one correspondence with points (x, y) of the open set $W = \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. This correspondence, denoted by $\tilde{\phi}$ is given by $\tilde{\phi}(X_p) = (\phi(p), y^1, \ldots, y^n)$, where $\phi(p) = (x^1, \ldots, x^n)$ are the coordinates of p, $X_p = \sum y^i \frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^i}$ are the coordinate frames. We have in fact defined an atlas on $T(M)$. This is $\{(\tilde{U}, \tilde{\phi})\}$, where $\tilde{U}$ is all X_p such that $p \in U$ and $\tilde{\phi}$ defined above. It remains to check that $T(M)$ is indeed a smooth manifold.

Check that if $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ are in the atlas then $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is smooth. (We should also check that the topology defined on M in the usual way by this atlas is Hausdorff and has a countable basis. This is entirely optional.)

7. Show that the Inverse Function Theorem is a special case of the general Implicit Function Theorem.

(HINT: Define $g : \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$g(x^1, \dots, x^n, y^1, \dots, y^n) = (y^1 - f^1(x^1, \dots, x^n), \dots, y^n - f^n(x^1, \dots, x^n)).$$

You are studying $M = g^{-1}(0, \dots, 0)$ at points x_0 where $det\frac{\partial y}{\partial x} \neq 0$. So if you can identify a suitable non-vanishing $n \times n$ subdeterminant of Dg, what does IFT tell you?)

8. Let M be a compact, n-dimensional, oriented manifold with boundary ∂M. Let ω be a (compactly supported) $(n-1)$-form on M and f a smooth function on M.

Prove that $\int_D f d\omega = \int_{\partial D} f\omega - \int_D df \wedge \omega$. (HINT: $d(f\omega) =$??)

9. Let f be a smooth function of one variable, let σ be the 2-form

$$\sigma = f(x^2 + y^2 + z^2)(x\, dy \wedge dx + y\, dz \wedge dx + z\, dx \wedge dy).$$

Compute the integral of σ over the sphere of radius r :

$$S_r = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = r^2\}$$

in terms of $f(r^2)$, provided S_r is oriented with outward normal.

10. Give examples of nonzero 2-forms ω and μ on $\mathbb{R}^4$, with $\omega \wedge \omega \neq 0$ and $\mu \wedge \mu = 0$.

11. Evaluate $\int_S \omega|_S$ where S is a helicoid in $\mathbb{R}^3$ parameterized by $\phi(s, t) = (s \cos t, s \sin t, t)$ for $s \in [0, 1]$, $t \in [0, 4\pi]$ and $\omega = z\, dx \wedge dy + 3\, dz \wedge dx - x\, dy \wedge dz$. (HINT: you'll need to use the map ϕ.)