Math 25a Homework 10

Due Tuesday 29th November 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) We know that the set $M_{2\times 2}(\mathbb{R})$ of 2×2 real matrices forms a vector space over $\mathbb{R}$ (with the usual definitions of addition and scalar multiplications for matrices).

(a) Show that the subspace W consisting of *symmetric* matrices is a subspace of $M_{2\times 2}(\mathbb{R})$. (Note: $W = \{A \in M_{2\times 2}(\mathbb{R}) : A = A^T\}$.)

(b) Find a basis for W. What is the dimension of W?

(c) Find a basis for $M_{2\times 2}(\mathbb{R})$ which contains a basis for W.

(2) Given a finite dimensional vector space V over a field F. We aim to show that $V^{**} \cong V$. Recall the handy bracket notation we discussed in class (Monday 21st November). We discussed the map $\langle , \rangle : V^* \times V \to F$ with $(f, v) \mapsto \langle f, v\rangle = f(v)$.

(a) Given any nonzero vector $v \in V$, show that there is a $f \in V^*$ such that $f(v) = 0$.

(b) Define a map $\phi : V \to V^{**}$ by $\phi(v) = \langle \cdot, v\rangle$. (That is, for $f \in V^*$, $\phi(v)(f) = \langle f, v\rangle = f(v)$.) Show that this map is injective (you might need part (a)). Then show the map is an isomorphism by counting dimensions (carefully).

Remark: Once you write out the (short) proof, it might seem that we haven't really done anything in part (b). In fact, the isomorphism we constructed is *canonical*. If we have time, we'll see precisely what this means in a future HW. For now, roughly, a canonical isomorphism means that it is independent of any choices. Contrast this with previous work. We (now) know that in fact $V^* \cong V$. This is because we constructed a basis for V^* from a basis of V — but note this isomorphism depended on the choices made.

(3) Recall from class (Monday 21st November), that we defined the annihilator of a subset of a vector space. Given a vector space V and subset S (note: S does not have to be a subspace of V!), then the annihilator $S^0 := \{f \in V^* : \langle f, s\rangle \equiv 0, \forall s \in S\}$.

(a) If W is an m-dimensional subspace of an n-dimensional vector space V, then show that W^0 is an $(n - m)$-dimensional subspace of V^*.

(b) Show that if W is a subspace of a finite dimensional vector space V, then $W^{00} = (W^0)^0 = W$.

(c) Show that if S is any subset of a finite-dimensional vector space V, then S^{00} is the same as the subspace of V spanned by vectors in S.

(d) If S and T are subspaces of a finite-dimensional vector space V and if $S \subset T$, then $T^0 \subset S^0$.

(e) If W and U are subspaces of a finite-dimensional vector space V, then $(W \cap U)^0 = W^0 + U^0$ and $(W + U)^0 = W^0 \cap U^0$.

(4) (a) Read Chapter 4 of Axler

(b) Problem 5 page 73 of Axler. (I can think of two different proofs - can you? Note: just submit one for grading.)

2 Ivan's problems

(1) *Exact Sequences*

A sequence of linear maps between vector spaces

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n$$

is called *exact* if and only if

$$im(T_i) = ker(T_{i+1}) \quad \text{for} \quad i = 1, 2, \ldots, n-2$$

For the rest of this question we write the zero vector space $\{0\}$ as 0.

(a) Show that

$$0 \longrightarrow V \xrightarrow{T} W \longrightarrow 0$$

is exact if and only if T is an isomorphism.

(b) Under what circumstances are

$$V \xrightarrow{T} W \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow V \xrightarrow{T} W$$

exact?

(c) Show that if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

is exact, then

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

(2) *Inverting matrices using elementary row and column operations*

An elementary row operation on a matrix is one of the following:

1. interchanging two rows
2. adding a multiple of one row to a different row
3. multiplying a row by a non-zero scalar.

Elementary column operations are the same but with "row" replaced with "column".

(a) Show that if matrices A and B are related by an elementary row operation then there exists an invertible matrix E such that $EA = B$. Show that if matrices A and B are related by an elementary column operation then there exists an invertible matrix E' such that $AE' = B$

(b) The *rank* of an $m \times n$ matrix A is the dimension of $\mathbb{R}^m$ spanned by the columns of A. Show that the rank of A is equal to the rank of the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ given by $v \mapsto Av$. Show that the rank of a matrix is left unchanged by elementary row and column operations.

(c) Find out what the "reduced row-echelon form" of a matrix is. Convince yourself that one can always put a matrix into reduced row-echelon form using a sequence of elementary row operations. Do a couple of small examples. (Note: no need to submit the small examples for grading.)

(d) For the remainder of the question, assume that A is an invertible $n \times n$ matrix, or in other words an $n \times n$ matrix of rank n. Deduce that one can find a sequence $E_1, \ldots E_N$ of matrices such that

$$E_N E_{N-1} \ldots E_1 A = I$$

where I is the $n \times n$ identity matrix and E_i are matrices of the type you constructed in part (a).

(e) Show that if one makes a $2n \times n$ matrix by writing A next to the identity matrix I:

$$B = \begin{pmatrix} A & I \end{pmatrix}$$

and then performs elementary row operation on B until A becomes the identity matrix, then the resulting matrix B' is

$$B' = \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

(f) Compute the inverse of the following matrix.

$$\begin{pmatrix} 1 & 6 & 5 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

(g) What happens if you try to use this method to compute the inverse of a non-invertible matrix A?

3 Warm up and Extra Problems.

This week, I'd suggest that the only extra problem we all face is how to eat all that turkey! (This is a tough one....)

4 Just for fun!

I'm getting a bit low on puzzlers for this semester so please feel free to submit a puzzler you think the class might enjoy. The puzzlers don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

This week, I have discovered a hilarious take on how to solve a quadratic equation. It is called "If the IRS has discovered the quadratic formula..." It is by Daniel J. Velleman. You get to it by going to his website
http://www.cs.amherst.edu/∼djv/
Then scrolling down to unpublished papers. It is the second link that you want. I think your whole family might enjoy this one, especially your parents!

By the way, his other unpublished paper is another take on the Fundamental Theorem of Algebra. I had a quick look. It is an OK read and you'll enjoy his take on the proof. Note that for some reason the pictures he claims is there don't seem to be there. (Well my version doesn't have them anyway.)