

Math 25a Homework 12

Due Tuesday 13th December 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Use induction to prove that $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$ for $n \geq 1$. Here M^n means M multiplied by itself n times and F_n is the n th Fibonacci number. That is, $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

(b) Find the eigenvalues of M and also an eigenvector for each eigenvalue. Express $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of these eigenvectors.

(c) Use your answers to (a) and (b) to show that for $n \geq 1$,

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right].$$

(2) Let W be a subspace of a vector space V , and let $T : V \rightarrow Z$ be a linear transformation. Let $\pi : V \rightarrow V/W$ be the natural map from V to V/W defined by $\pi(v) = [v]$. (Recall that $[v]$ is the equivalence class of v in V/W and $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$.) Show that there exists a linear transformation $\bar{T} : V/W \rightarrow Z$ such that $T = \bar{T} \circ \pi$ if and only if $W \subset \ker(T)$.

Problems (3), (4) and (5) will be dedicated to understanding the tensor product $V \otimes W$ of two vector spaces V and W . (Note that these problems will be similar in style to the ones that you did on the product space $\prod V_i$ of vector spaces V_i in HW 11.)

Given vector spaces V and W over a field F we define the tensor product $V \otimes W$ to be a vector space over F together with a bilinear map¹ $\mu : V \times W \rightarrow V \otimes W$ satisfying the following universal mapping property: Given any vector space U and bilinear map $f : V \times W \rightarrow U$, there is a unique linear transformation $\hat{f} : V \otimes W \rightarrow U$ such that $\hat{f} \circ \mu = f$.

In problem (3) we'll construct $V \otimes W$ and in problem (4) show it satisfies the universal mapping property for tensors. Finally in problem (5) we'll show that the UMP for tensors defines the tensor product uniquely up to isomorphism.

(3) Let $X = \coprod_{i \in I} F_i$, where the indexing set $I = V \times W$ and $F_i = F$ (so F is viewed as a one-dimensional vector space over F). Recall from HW 11 that this means

$$X = \{g : V \times W \rightarrow F \mid g((v, w)) = 0 \text{ for all but finitely many choices of } (v, w) \in V \times W\}.$$

(a) If we define $\delta_{(v,w)} \in X$ by

$$\delta_{(v,w)}(v', w') = \begin{cases} 1 & \text{if } (v', w') = (v, w), \\ 0 & \text{otherwise,} \end{cases}$$

then show that $\{\delta_{(v,w)} \mid (v, w) \in V \times W\}$ is a basis for X over F .

Let

$$Z = \text{span} \left(\{\delta_{(a_1 v_1 + a_2 v_2, w)} - a_1 \delta_{(v_1, w)} - a_2 \delta_{(v_2, w)}\} \cup \{\delta_{(v, b_1 w_1 + b_2 w_2)} - b_1 \delta_{(v, w_1)} - b_2 \delta_{(v, w_2)}\} \right)$$

for all $a_1, a_2, b_1, b_2 \in F$, all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. Note that Z is the span of an infinite set² and note that Z is a subspace of X .

Define $T := X/Z$ (a quotient of vector spaces) and $\mu : V \times W \rightarrow T$ by $\mu((v, w)) = [\delta_{(v,w)}]$ (the equivalence class of $\delta_{(v,w)}$ in the quotient).

(b) Check that μ is indeed bilinear.

(4) Show that $(T, \mu : V \times W \rightarrow T)$ satisfies the universal mapping property for tensor products. (Here T and μ are defined as in problem (3).)

HINT 1: This means for any vector space U and any bilinear map $f : V \times W \rightarrow U$ there exists a unique map $\hat{f} : T \rightarrow U$ such that $f = \hat{f} \circ \mu$. So we have to construct such an \hat{f} , show it is linear, unique and show that $f = \hat{f} \circ \mu$.

¹Recall that a bilinear map $f : V \times W \rightarrow U$ is a map which is linear in each slot. That is $f(a_1 v_1 + a_2 v_2, w) = a_1 f(v_1, w) + a_2 f(v_2, w)$ and $f(v, b_1 w_1 + b_2 w_2) = b_1 f(v, w_1) + b_2 f(v, w_2)$ for all $a_1, a_2, b_1, b_2 \in F$, all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$.

²Recall that $\text{span}(Y)$ where Y is infinite is defined to be the set of all *finite* linear combinations of elements in Y .

HINT 2: To construct such an \hat{f} , we'll first define a linear map $\bar{f} : X \rightarrow U$ by

$$\bar{f}(g) := \sum_{(v,w) \in V \times W} g((v,w))f((v,w)).$$

Note that this is a finite sum since $g \in X$. Now check that $\bar{f}(Z) = \{0\}$. Use problem (2) to conclude that we have a well-defined linear map $\hat{f} : T \rightarrow U$ given by $\hat{f}([g]) = \bar{f}(g)$.

(5) Suppose that $(T', \mu' : V \times W \rightarrow T')$ is another pair satisfying the universal mapping property of a tensor product. Show that $T \cong T'$. (HINT: This is exactly the same style of argument as for coproducts and products that you saw in HW 11.)

FINAL REMARKS: The tensor product of V and W is usually denoted $V \otimes W$ and the image $\mu(\delta_{(v,w)}) = [\delta_{(v,w)}]$ is generally written as $v \otimes w$. Make sure that you understand that in this notation $(a_1v_1 + a_2v_2) \otimes w = a_1(v_1 \otimes w) + a_2(v_2 \otimes w)$ and $v \otimes (b_1w_1 + b_2w_2) = b_1(v \otimes w_1) + b_2(v \otimes w_2)$.

2 Ivan's problems

- (1) (a) Problem 24 on page 125 of Axler. (Hint: use your answer to question 10.)
(b) Problem 27 on page 125 of Axler.
- (2) (a) Problem 28 on page 125 of Axler.
(b) Problem 29 on page 125 of Axler.
- (3) (a) Problem 30 on page 125 of Axler.
(b) Problem 31 on page 125 of Axler.
- (4) (a) Problem 2 on page 158 Axler.
(b) Problem 4 on page 158 Axler.
- (5) (a) Problem 9 on page 159 Axler.
(b) Problem 10 on page 159 Axler.

3 Warm up and Extra Problems.

Lots of great extra problems. Take a look at Axler Ch 6 Problems 21, 22, 23 and 25. Take a look at Axler Ch 7 Problems 1, 3, 5, 6, 7, 11 and 12.