Math 25a Homework 13

Due Tuesday 20th December 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) Problem 12 on page 245 of Axler.

(2) Problem 9 on page 244 of Axler.

(3) (a) Problem 14 on page 245 of Axler.
(b) Problem 15 on page 245 of Axler.

(4) (a) Problem 16 on page 245 of Axler.
(b) Problem 19 on page 245 of Axler.

(5) (a) Problem 20 on page 245 of Axler.
(b) Problem 21 on page 245 of Axler.

2 Ivan's problems

(1) (a) Problem 6 on page 122 of Axler (so back to Ch 6).
(b) Take a look at Problem 7 on page 122 of Axler. Don't hand this one in. The computations are similar, but longer than Problem 6. I meant to set these problems earlier. These reformulations of the inner product can be quite handy.

(2) Recall from HW 12 that we defined the tensor product of two vector spaces. Stop right now, get out your old assignment and take a look at all the notation we covered. I remarked at the end of Problem 5, that the tensor product of V and W is usually denoted $V \otimes W$ and the image $\mu((v,w)) = [\delta_{(v,w)}]$ is generally written as $v \otimes w$. Make sure that you understand that in this notation $(a_1 v_1 + a_2 v_2) \otimes w = a_1(v_1 \otimes w) + a_2(v_2 \otimes w)$ and

$v \otimes (b_1 w_1 + b_2 w_2) = b_1(v \otimes w_1) + b_2(v \otimes w_2)$. (In case you wondered about the peculiar definition of Z in the previous HW, note that it is this very definition that gives tensors these nice properties.)

Now let V and W be vectors spaces over a field F and let V have basis $R = \{v_1, \ldots, v_n\}$ and let W have basis $S = \{w_1, \ldots w_m\}$. The aim of this question is to show that $B = \{v_i \otimes w_j | 1 \le i \le n, 1 \le j \le m\}$ is a basis for $V \otimes W$.

(a) Show that B spans $V \otimes W$.

(b)(i) Let

$$U = \coprod_{R \times S} F.$$

Show that $\{f_{ij} : R \times S \to F\}$ is a basis for U where f_{ij} is defined as follows:

$$f_{ij}((v_k, w_l)) = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

(b)(ii) Show that the map $\phi : V \times W \to U$ defined by $\phi(\sum_i a_i v_i, \sum_j b_j w_j) := \sum_{i,j} a_i b_j f_{ij}$ is bilinear.

(b)(iii) Show that $B = \{v_i \otimes w_j | 1 \le i \le n, 1 \le j \le m\}$ is a linearly independent set in $V \otimes W$.

(HINT: By the Universal Mapping Property for tensors, there exists a unique linear map $\hat{\phi} : V \otimes W \to U$ such that $\hat{\phi} \circ \mu = \phi$. (Recall that $\mu : V \times W \to V \otimes W$ was defined in HW 12.) Now suppose that $\sum a_{ij} v_i \otimes w_j = 0$. Apply $\hat{\phi}$ to both sides.)

(3) (a) If V is a finite dimensional space, and if u and v are in V, it is true that $u \otimes v = v \otimes u$?

(b) Now let x, y be a basis for the vector space V over the field F. Using Problem 2 we know that every element of $V \otimes V$ can be written uniquely in the form

$$ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y$$

for some $a, b, c, d \in F$. Firstly, show that $x \otimes y + y \otimes x$ is not equal to $v \otimes w$ for any choice of $v, w \in V$. Secondly, for what values of a, b, c, d can the tensor $ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y$ be written as $v \otimes w$?

(4) Suppose that V and W are finite dimensional vector spaces over a field F. Recall that we proved that the set

$$\mathcal{L}(V, W) = \{T : V \to W : T \text{ is linear}\}$$

forms a vector space. (See Axler page 40.) Construct an isomorphism between $\mathcal{L}(V, W)$ and $V^* \otimes W$ where V^* is the dual space to V. (In fact, this isomorphism is canonical — it will not depend on the bases chosen.) (Hint: as useful as it is, don't just count dimensions. I want to see an actual map. There are at least two ways to do this problem. One of these ways uses the knowledge you've gained from Problem 2.)

3 Holiday Puzzler

There is one last puzzler for the semester. I've given you quite a challenging one that might take you the break to work out.

Alison, Ivan and I play a game together. Alison and Ivan each think of a positive integer. They each tell me their number. On the blackboard I then write down two numbers. One of them is the sum of Alison's and Ivan's numbers, the other is a positive integer chosen at random. I don't tell Alison and Ivan which is which. I then ask Alison if she knows what Ivan's number is. If she knows it she tells me yes, if she can't figure it out, she tells me no. If she says no, I then ask Ivan if he knows what Alison's number is. If he can figure it out, he says yes and if he can't figure it out he says no. If Ivan says no, I then ask Alison if she knows what Ivan's number is. She then says yes or no depending on whether or not she's figured it out and so the game continues.

Now, as we all know, Alison and Ivan are excellent mathematicians and logicians and they are always honest!! So they always truthfully reply when giving me their answers. Can you prove that this process of questioning will terminate. That is, that eventually one of them will know the other's number.