Math 25a Homework 6

Due Tuesday 1st November 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) Problem 6 on page 78 of Rudin.

(2) Problem 7 on page 78 of Rudin.

(3) Problem 8 on page 79 of Rudin.

(4) Problem 10 on page 79 of Rudin.

2 Ivan's problems

(1) Problem 14 on page 80 of Rudin.

(2) Let S be the set of positive integers that do not involve the digit 0 in their decimal representation (so, for example, $17 \in S$ but $107 \notin S$). Show that

$$\sum_{n \in S} \frac{1}{n}$$

converges to a value less than 90.

(3) Tom Coates told me this problem which originally came from Dick Gross, dean of Harvard College.

(a) Let

$$s(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Show that this power series converges for $x \in (-1, 1]$. Give an argument to show that

$$s'(x) = \frac{1}{1+x^2}$$

and hence that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}.$$

(Do not worry about making the latter part of your argument rigorous. We will deal with differentiation of power series etc. later in the course in Spring semester.)

(b) Let

$$a_1 = 1 \quad a_2 = -1/3 \quad a_3 = 1/5 \quad a_4 = -1/7 \quad \ldots$$

and consider the partial sums

$$s_1 = a_1 \quad s_2 = a_1 + a_2 \quad s_3 = a_1 + a_2 + a_3 \quad \ldots$$

Show that

$$s_2 < s_4 < s_6 < \cdots < \frac{\pi}{4} < \cdots < s_5 < s_3 < s_1$$

(c) Let

$$t_1 = \frac{s_1 + s_2}{2} \quad t_2 = \frac{s_2 + s_3}{2} \quad t_3 = \frac{s_3 + s_4}{2} \quad \ldots$$

Show that

$$t_2 < t_4 < t_6 < \cdots < \frac{\pi}{4} < \cdots < t_5 < t_3 < t_1$$

(d) Let

$$u_1 = \frac{t_1 + t_2}{2} \quad u_2 = \frac{t_2 + t_3}{2} \quad u_3 = \frac{t_3 + t_4}{2} \quad \ldots$$

Show that

$$u_2 < u_4 < u_6 < \cdots < \frac{\pi}{4} < \cdots < u_5 < u_3 < u_1$$

(e) Use part (d) and the first 10 terms of the sequence a_n to give an estimate for π.

3 Warm up problems

(1) Problem 9 page 79 Rudin (power series examples).

4 Supplementary Problems

(1) If you haven't already thought about this, take a look at Problem 19 page 81. This gets you to think about the Cantor set in terms of decimal expansions in base 3.

5 The Math Puzzler - just for fun!

Each week there will be a "math puzzler" for the class to think about. Please feel free to submit a "puzzler" you think the class might enjoy. The "puzzlers" don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

This problem is from Math Horizons Sept 2005 issue. It is a great logic puzzle from a book by Raymond Smullyan.

The Island of Dreamers. On the Island of Dreamers there are two kinds of people — diurnal and nocturnal. Diurnals are characterized by this property: everything they believe while awake is true while everything they believe while asleep is false. Conversely, everything nocturnals believe while sleeping is true and everything nocturnals believe while awake is false.

(1) An inhabitant of the Island of Dreamers once believed he was diurnal. Can you determine if his belief was correct? Can you determine if he was awake or asleep?

(2) Similarly, another inhabitant believed she was asleep. Can you determine if her belief was correct? Can you determine whether she is nocturnal or diurnal?

HINT: there are four laws that help in solving problems about the Island of Dreamers — can you prove them?

(I) Any inhabitant awake believes he is diurnal
(II) Any inhabitant asleep believes he is nocturnal
(III) A diurnal inhabitant always believes he is awake
(IV) A nocturnal inhabitant always believes he is asleep.