

Math 25a Homework 9

Due Tuesday 22nd November 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) Problem 1 on page 59 of Axler

(2) (a) Problem 9 on page 59 of Axler.

(b) Let \mathcal{P}_n be the set of all real polynomials of degree less than n . Consider the linear transformations $S : \mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$ such that $S(f) = \int_0^x f(t)dt$ and $T : \mathcal{P}_{n+1} \rightarrow \mathcal{P}_n$ such that $T(g) = g'$.

(i) Compute the matrix A of S with respect to the standard bases for \mathcal{P}_n and \mathcal{P}_{n+1} .

(ii) Compute the matrix B of T with respect to the standard bases for \mathcal{P}_n and \mathcal{P}_{n+1} .

(iii) Compute the matrix products AB and BA . What do you observe — any suggestions as to why this happens?

(3) Let V be a finite dimensional vector space over a field F and let v_1, \dots, v_n be a basis for V .

(a) If $a_1, \dots, a_n \in F$, then prove that there is a **unique** $f \in V^*$ such that $f(v_1) = a_1$, $f(v_2) = a_2, \dots, f(v_n) = a_n$. (Recall V^* is the dual space of V .)

(b) Using part (a), define $f^i : V \rightarrow F$ by

$$f^i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Prove that $\{f^1, f^2, \dots, f^n\}$ is a basis for V^* .

(4) Problem 11 on page 60 of Axler. (Warning: you can't use Theorem 3.4 of Axler.)

(5) Problem 17 on page 60 of Axler.

(6) Problem 22 on page 61 of Axler.

2 Ivan's problems

(1) (a) Let V be a vector space and let U and W be subspaces of V such that $U \oplus W = V$. Define a map $\phi : U \rightarrow V/W$ such that $\phi(u) = [u]$. Show that ϕ is an isomorphism. (That is show ϕ is linear, injective and surjective). Hence $U \cong V/W$.

(Hint: Recall that V/W is the quotient space of V with W , with $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$ and the equivalence class of v is denoted by $[v]$ — see HW 8.)

(b) If V is a finite dimensional vector space, W a subspace of V with $\dim V = n$ and $\dim W = m$, then using part (a) or otherwise, prove that $\dim V/W = n - m$.

(2) Explain how and why a linear transformation $T : V \rightarrow W$ induces an isomorphism between $V/\ker(T)$ and $\text{im}(T)$.

(3) Problem 13 on page 60 of Axler.

(4) Problem 14 on page 60 of Axler. (Hint: you might need problem 3 on page 59 of Axler.)

(5) Problem 16 on page 60 of Axler.

(6) Problem 26 on page 61 of Axler.

3 Warm up and Extra Problems.

As we've finally started talking about matrices, the first three chapters of Otto Bretscher's text will make some sense! If you're looking for some calculations to try, I'd suggest trying just about anything from Bretscher as well as Problems 2 and 10 on page 59-60 of Axler.

As usual any of the problems in Axler provide great practice for the material. In particular, Problems 13, 15, 18, 23 and 24 page 60-61 go very nicely with the HW problems set for this week. You might also consider the following two problems.

(1) Let $L \subset \mathbb{R}^2$ be the line $y = mx$. Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which maps the point (x, y) to the point $(p, q) \in L$ which is closest to (x, y) . Compute the matrix of T with respect to a basis of \mathbb{R}^2 that makes this problem easy. Now compute the matrix of T with respect to the standard basis of \mathbb{R}^2 . (Note: you're smart people, so as you find your way through this problem you'll figure out how a change of basis affects a matrix!)

(2) Let \mathcal{P} be the vector space of all polynomials with coefficients in \mathbb{R} . We'll consider \mathcal{P}/M for a subspace M of \mathcal{P} . For which of the following cases is \mathcal{P}/M finite dimensional?

(a) $M = \mathcal{P}_m$ be the subspace of all polynomials of degree less than m .

(b) M is the subspace consisting of all even polynomials.

(c) M is the subspace consisting of all polynomials divisible by x^n .

4 The Math Puzzler — Just for fun!

The puzzler is back! I'm getting a bit low on puzzlers for this semester so please feel free to submit a puzzler you think the class might enjoy. The puzzlers don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

This week's puzzler is such a classic problem in conditional probability that I don't have a reference for it. This math problem is based on a real life situation. **Warning: it has generated hours of fierce arguments amongst people! Friendships have been won and lost during these disputes.** Do you want to get involved? Read on.....

There is (was?) a game show (in Europe? in the 70s?) where the contestant is shown 3 doors. The prize (new car, jewelry, etc) lies behind one of the doors. The contestant picks a door. The game show host does not immediately open this door, rather, he opens (at random) one of the remaining two doors. If the prize is behind the open door, the contestant loses. If the prize is not behind the open door, the contestant has the opportunity to stay with the door they originally picked or to switch to the remaining door. The game show host then opens the door the contestant has chosen to reveal what lies behind it. What is the best strategy in this game? That is, should the contestant switch doors or should they stay with their original choice?