Name: ______________________________

MATH 134
Elizabeth Denne

Due: 10/1/2004

Worksheet 1

1. We write $\mathbf{R}^{n+1} = \{(x^1, x^2, \ldots, x^n) \,|\, x^i \in \mathbf{R}\}$. Now, $\mathbf{R}P^n$ = all unoriented lines through $0 = (0, \ldots, 0)$ in $\mathbf{R}^{n+1}$. Put $n+1$ charts on $\mathbf{R}P^n$ as follows: for $i = 1, 2, \ldots, n+1$, let U_i = all lines though 0 that intersect the plane $x^i = 1$. For a line $L \in U_i$, let

$$\phi_i(L) = (u_i^1(L), \ldots, u_i^{i-1}(L), u_i^{i+1}(L), \ldots, u_i^n(L))$$

be the other n coordinates of the intersection point. Show that $\{(U_i, \phi_i)\}$ is an atlas on $\mathbf{R}P^n$ as follows:
(HINT: try all of this first for $\mathbf{R}P^2$ and/or $\mathbf{R}P^3$.)

a) Exactly which lines through 0 do **not** lie in U_1?

b) Show that every line though 0 is in U_i for some i.

c) Show that $\phi_1(U_1 \cap U_2)$ is open in $\mathbf{R}^n$. (The others are similar.)
Specifically, $\phi_1(U_1 \cap U_2) = ??$

d) On $U_1 \cap U_2$ show that $\phi_2 \circ \phi_1^{-1}$ is smooth. That is, show u_2^j $j = 1, \ldots, n$ are smooth functions of u_1^j for $j = 1, \ldots, n$. Specifically, $u_2^1 = ?$, $u_2^2 = ?$,..., $u_2^n = ?$.

e) Now show $\phi_j \circ \phi_k^{-1}$ is smooth on $U_j \cap U_k$. (Hint: first assume $j < k$; $k < j$ is similar.)

2.a) Show that the union of the two coordinate axes in $\mathbf{R}^2$ is not a manifold.

b) Show that the set $\{(x, y) \in \mathbf{R}^2 | xy = a\}$ is a manifold if and only if $a \neq 0$. Construct an explicit atlas in the case when $a \neq 0$.

3. (Challenge) Show that the k-dimensional sphere cannot be made into a manifold where the differential structure is generated by an atlas with a single chart.
(HINT: S^k is compact.)