

## Worksheet 2

1.  $\mathbf{R}P^2 =$  all lines through 0 in  $\mathbf{R}^3$ . Put an atlas with 3 charts on  $\mathbf{R}P^2$  as in Worksheet 1:  $(U_1, \phi_1 = (u_1^1, u_1^2))$ ,  $(U_2, \phi_2 = (u_2^1, u_2^2))$ ,  $(U_3, \phi_3 = (u_3^1, u_3^2))$ . Recall that we showed  $u_2^1 = \frac{1}{u_1^1}$  and  $u_2^2 = \frac{u_1^2}{u_1^1}$ .

a) Calculate the determinant of  $D(\phi_2 \circ \phi_1^{-1}) = \frac{\partial(u_2^1, u_2^2)}{\partial(u_1^1, u_1^2)}$

b) Carefully show on the sketch (above right), and describe in words, the points on the upper hemisphere that lie on lines in  $U_1 \cap U_2$ .

c) Discuss the sign of the determinant of  $D(\phi_2 \circ \phi_1^{-1})$  for  $U_1 \cap U_2$ .

d) By switching the coordinates of one (or more) of your charts, can you make all  $\det D(\phi_j \circ \phi_i^{-1}) > 0$ ? Explain, using c). What does this say about the orientability of  $\mathbf{R}P^2$ ?

2.  $\mathbf{R}P^3 =$  all lines through 0 in  $\mathbf{R}^4$ . Put 4 charts on  $\mathbf{R}P^3$  as in Worksheet 1: For  $1 \leq i \leq 4$ , let  $U_i =$  all lines through 0 that intersect the 3-plane  $x^i = 1$ . As before, let  $(u_i^1, u_i^2, u_i^3)$  be the other 3 coordinates of the intersection point.

a) For  $k = 2, 3, 4$ , write  $u_k^1, u_k^2, u_k^3$  as functions of  $u_1^1, u_1^2, u_1^3$ .

b) Calculate  $\det D(\phi_k \circ \phi_1^{-1}) = \det \frac{\partial(u_k^1, u_k^2, u_k^3)}{\partial(u_1^1, u_1^2, u_1^3)}$  for  $U_1 \cap U_k$ :

On  $U_1 \cap U_2$ :

On  $U_1 \cap U_3$ :

On  $U_1 \cap U_4$ :

Please give some explanation/details.

c) Show that you can make all  $\det D(\phi_k \circ \phi_1^{-1}) > 0$ , by switching a pair of coordinates in one or more of your charts.

d) Note that  $D(\phi_j \circ \phi_i^{-1}) = D(\phi_j \circ \phi_1^{-1})D(\phi_1 \circ \phi_i^{-1}) = D(\phi_j \circ \phi_1^{-1})D(\phi_i \circ \phi_1^{-1})^{-1}$ . Use this to show that you can make  $\det D(\phi_j \circ \phi_i^{-1}) > 0$  on  $U_i \cap U_j \cap U_1$ , for all  $i, j$ .

But in  $\mathbf{R}P^3$  (as opposed to  $\mathbf{R}P^2!$ ), the sign of  $\det D(\phi_j \circ \phi_i^{-1})$  is the same on all the components of  $U_i \cap U_j$ , so you have made all  $\det D(\phi_j \circ \phi_i^{-1}) > 0$  everywhere....  
**CONGRATULATIONS!!!** You have just proved that  $\mathbf{R}P^3$  is **orientable**.

- e) You have enough information now to make a good guess:  $\mathbf{R}P^n$  is  
orientable if  $n$  is ???  
not orientable if  $n$  is ???