Worksheet 7

Elizabeth Denne Math 134

DUE: Friday November 19, 2004.

Please do the following problem.

1. Let M be an m-dimensional manifold. Let g be a smooth function on M. Let the coordinate charts on M be $(U, \phi = (x^1, \ldots, x^m))$. Then $dg : T_pM \to \mathbb{R}^n$ is defined $dg(\overrightarrow{v}_p) = \overrightarrow{v}_p(g) = \Sigma_{i=1}^m a_i \frac{\partial g}{\partial x^i}$, where $\overrightarrow{v}_p = \Sigma_{i=1}^m a_i \frac{\partial}{\partial x^i}$. Given an n-dimensional manifold N and a smooth map $F : M \to N$, show that $F^*(dg) = d(F^*g)$. (Here $F^*g = g \circ F$ and $F^*(dg)$ is the pull back of the covector (or 1-form) dg.)

Please also do at least one of the following problems.

2. Let $A : V \to W$ be a linear map between two vector spaces. Then for $T \in \bigwedge^p(W)$, we define $A^*T \in \bigwedge^p(V)$ by $A^*T(v_1, \ldots, v_p) = T(Av_1, \ldots, Av_p)$.
Please check (a) $A^*(aT_1 + bT_2) = aA^*(T_1) + bA^*(T_2)$ (where $a, b \in \mathbb{R}$) and
(b) $A^*(T \wedge S) = A^*(T) \wedge A^*(S)$.

3. Given an m-dimensional manifold M and two coordinate charts $(U, (x^1, \ldots, x^n))$ and $(U, (x'^1, \ldots, x'^n))$. On $U \cap U'$:
(a) Express $\frac{\partial}{\partial x'^i}$ in terms of the $\frac{\partial}{\partial x^j}$.
(b) For a vector field $v^k \frac{\partial}{\partial x^k} = v'^i \frac{\partial}{\partial x'^k}$, express v'^i in terms of the v^k.
(c) For a 1-form $v_k dx^k = v'_i dx'^i$, express v'_i in terms of the v_k.
(d) Differentiating both sides of your answers in (c) by $\frac{\partial}{\partial x'^j}$, express $\frac{\partial v'_i}{\partial x'^j}$ (call these "v'_{ij}") in terms of the function $\frac{\partial v_k}{\partial x^l}$ (call these "v_{kl}").
(e) Explain why this is NOT the transformation law of a tensor.

4. For an n-dimensional manifold M, define a $(2n)$-dimensional manifold:
T^*M = all covectors (or 1-forms) at all $p \in M$ = "cotangent bundle of M".
For any coordinate chart $(U, (x^1, \ldots x^n))$ on M, define a coordinate chart $(T^*U, (x^1, \ldots, x^n, y_1, \ldots, y_n)$ on T^*M by: T^*U = all covectors α_p, for all $p \in U$ = all $\Sigma y_i dx^i_{|p}$ for all $p \in U$.
(a) On $T^*U \cap T^*U'$, find the Jacobian matrix $\frac{\partial(x'^1, \ldots, x'^n, y'_1, \ldots, y'_n)}{\partial(x^1 \ldots, x^n, y_1, \ldots, y^n)}$.

(b) Show that this Jacobian has positive determinant. What does this tell you about T^*M?
(c) Show there is a 1-form θ on T^*M, given in coordinates by $\theta = \Sigma_{i=1}^n y_i dx^i$. (Hint: just show that on $T^*U \cap T^*U'$, $\Sigma_{i=1}^n y_i dx^i = \Sigma_{i=1}^n y_i' dx'^i$.)
(d) The 2-form $\omega = d\theta$ is given in coordinates by $\omega = \Sigma_{i=1}^n dy_i \wedge dx^i$. Show that $\omega^n = \omega \wedge \ldots \wedge \omega$ is NEVER zero. (This means that T^*M carries a natural volume form $2n$-form. There is no such form on TM, unless we choose a Riemannian metric on M.)

5. For $M = \mathbb{R}^2$, let $\omega = (x^2 + y^2) dx \wedge dy$.
(a) Express ω in polar coordinates (r, θ). Please show your work. (Hint: write x and y in terms of r and θ. Then calculate dx, dy and then ω by substitution.)
(b) From our discussions about volume forms, you know that if M is an n-dimensional manifold, and ω is an n-form on M, then under coordinate change:

$$\begin{aligned} \omega &= f(x^1, \ldots, x^n) dx^1 \wedge dx^n \\ &= f(x^1(x'^1, \ldots, x'^n), \ldots, x^n(x'^1, \ldots, x'^n)) det \frac{\partial(x^1, \ldots, x^n)}{\partial(x'^1, \ldots, x'^n)} dx'^1 \wedge dx'^n. \end{aligned}$$

Check your answer in (a) agrees with this.