Worksheet 8

Elizabeth Denne Math 134

DUE: Friday December 3, 2004.

Please do questions 1 through 4. (They are all related.) Note that Q2(c) is optional.

1. The map $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $F(u^1, u^2, u^3) = (x^1, x^2, x^3) = (u^1 \cos(2\pi u^2), u^1 \sin(2\pi u^2), u^3)$.

(a) Sketch the image under F of the unit cube $[0,1]^3 = \{(u^1, u^2, u^3) : 0 \leq u^i \leq 1\}$.

For the 2-form $\alpha = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$ find:

(b) $d\alpha$

(c) $F^* d\alpha$

(d) $F^* \alpha$

(e) $dF^* \alpha$

2. Let $\gamma : [0,1]^3 \to M$ by a singular 3-cube.

(a) In the picture below, label the six 2-cubes indicated by $\gamma_{i,\alpha}$, where $i = 1, 2, 3$ and $\alpha = 0, 1$.

(b) $\partial\gamma =$??

(c) **This part is optional.** Label the 12 1-cubes (oriented for increasing u^i) and compute $\partial(\partial\gamma) = \partial^2\gamma$.

3. Let $\alpha = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$ be a 2-form on $\mathbb{R}^3$. Let $\gamma : [0,1]^3 \to \mathbb{R}^3$ be given by $x^1 = u^1 \cos(2\pi u^2)$, $x^2 = u^1 \sin(2\pi u^2)$, $x^3 = u^3$.

(a) Calculate $\int_\gamma d\alpha$ **from the definition**. (Hint: Q1(c).)

(b) From your sketch in Q1 (a) above, why should you expect this answer? (I'm looking for a few sentences here, not an essay.)

(c) $d\gamma = \gamma_{1,1} + ???$ (Hint: Q2(b).)

Now find $\gamma_{1,1}(u^2, u^3) = ???$ (That is, write out an expression for the map.)

Do this for all $\gamma_{i,\alpha}$, $i = 1, 2, 3$, $\alpha = 0, 1$. Can you simplify the expression for $d\gamma$?

4. Calculate $\int_{\partial\gamma} \alpha$ **from the definition**.

The following two problems are optional, but contain a neat application of everything we are doing. Take the time to read this through carefully.

5. Let M be an 2-dimensional manifold. Recall that a 2-dim manifold is orientable if it has an atlas $\{U, \phi\}$ such that on all coordinate charts that overlap $U \cap U'$, $\det \frac{\partial(u^1,u^2)}{\partial(u'^1,u'^2)} > 0$.

(a) Let $U, \phi = (u^1, u^2)$ and $U', \phi' = (u'^1, u'^2)$ be two coordinate charts on M, such that $U \cap U' \neq \emptyset$. On $U \cap U'$, write $\frac{\partial}{\partial u'^1}$ and $\frac{\partial}{\partial u'^2}$ in terms of $\frac{\partial}{\partial u^1}$ and $\frac{\partial}{\partial u^2}$.

(b) Suppose M is a submanifold of $\mathbb{R}^3$ (so by using F_* we can just think of the vectors in (a) as vectors in $\mathbb{R}^3$). Show the definition of orientability given in Q5 is the same as saying M has an atlas such that on every $U \cap U'$, $\frac{\partial}{\partial u'^1} \times \frac{\partial}{\partial u'^2}$ is a **positive** scalar multiple of $\frac{\partial}{\partial u^1} \times \frac{\partial}{\partial u^2}$.

6. Now $\alpha = a_1(x^1,x^2,x^3)\, dx^2 \wedge dx^3 + a_2(x^1,x^2,x^3)\, dx^3 \wedge dx^1 + a_3(x^1,x^2,x^3)\, dx^1 \wedge dx^2$ is a 2-form on $\mathbb{R}^3$. Let M be a surface in $\mathbb{R}^3$, with a coordinate chart $(U, \phi = (u^1, u^2))$. So we have a map $F : M \to \mathbb{R}^3$, given in this coordinate chart by $x^i = x^i(u^1, u^2)$, $i = 1, 2, 3$.

(a) Find $F^*\alpha$.

(b) Express your answer in terms of $F_* \frac{\partial}{\partial u^1} \times F_* \frac{\partial}{\partial u^2}$.

Conclusion: If M is an oriented surface, $F_* \frac{\partial}{\partial u^1} \times F_* \frac{\partial}{\partial u^2} = ||F_* \frac{\partial}{\partial u^1} \times F_* \frac{\partial}{\partial u^2}|| \overrightarrow{N}$ where $\overrightarrow{N}$ is a continuous unit normal vector which is independent of the choice of coordinate chart (U, ϕ). (We checked this in Q5.) Now $||F_* \frac{\partial}{\partial u^1} \times F_* \frac{\partial}{\partial u^2}||$ is the area in T_pM of the parallelogram spanned by $F_* \frac{\partial}{\partial u^1}$ and $F_* \frac{\partial}{\partial u^2}$. Thus the **area 2-form** on M is $||F_* \frac{\partial}{\partial u^1} \times F_* \frac{\partial}{\partial u^2}||\, du^1 \wedge du^2$ (for the Riemannian metric on M induced by the usual dot product on $\mathbb{R}^3$).