Worksheet 9

Elizabeth Denne Math 134

DUE: Friday December 10, 2004.

Please read through all questions.
Please do at least one question from questions 1, 2 and 3.

1. Let $M = \mathbb{R}^3$ and take a 2-form $\alpha = v_1\, dx^2 \wedge dx^3 + v_2\, dx^3 \wedge dx^1 + v_3\, dx^1 \wedge dx^2$ (where $v_i = v_i(x^1, x^2, x^3)$). Find $d\alpha$.

2. If α is a 0-form on M (so $\alpha = f(x^1, \ldots, x^n)$ in a coordinate chart), calculate $d(d\alpha)$ and show it is the **zero** 2-form.

3. Let $\alpha = a\, dx^I = a(x^1 \ldots, x^n)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$ and $\beta = b\, dx^J = b(x^1 \ldots, x^n)\, dx^{j_1} \wedge \ldots \wedge dx^{j_r}$ (so α is a r-form and β is an s-from and there is no Einstein summation here). Express $d(\alpha \wedge \beta)$ in terms of $(d\alpha) \wedge \beta$ and $\alpha \wedge (d\beta)$. Please don't quote the result - write it out! The following calculations will be helpful:

a) $\alpha \wedge \beta = ab\, dx^I \wedge dx^J$
b) $(d\alpha) \wedge \beta = ??$
c) $\alpha \wedge d\beta = ??$
d) $d(\alpha \wedge \beta) = ??$

Please then do at least two questions from questions 4, 5, 6 and 7.

4. Show that $\mathbb{R}^n$ is smoothly contractible to a point: Define a smooth map $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ with $F(x^1, \ldots, x^n, 0) = (x^1, \ldots, x^n)$ and $F(x^1, \ldots, x^n, 1) = (0, \ldots, 0)$:

$$F(x^1, \ldots, x^n, t) = ?? \qquad\qquad x^i \in \mathbb{R},\ 0 \le t \le 1.$$

5. On $M = \mathbb{R}^4$, let $\alpha = x^2\, dx^1 \wedge dx^3 \wedge dx^4 + x^1\, dx^2 \wedge dx^3 \wedge dx^4 + x^3\, dx^2 \wedge dx^4 \wedge dx^1 + x^2\, dx^3 \wedge dx^4 \wedge dx^1$.

a) Show α is closed.
b) Show $\int_\gamma \alpha = 0$ for any singular 3-chain γ with $\partial\gamma = 0$. (HINT: what does Q4 tell you about α?)

6. Let $\alpha = x\,dy - y\,dx$ be a 1-form on $\mathbb{R}^2 \setminus \{0\}$. Compute the integral of α over the circle $S = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 4\}$, where S is oriented counterclockwise. (HINT: there is a really fast way to do this with minimum computation.)

7. Let $M = \mathbb{R}^3$ and $\alpha = \Sigma a_i\, dx^i$ be a 1-form on $\mathbb{R}^3$ (where $a_i = a_i(x^1, x^2, x^3)$). Let $\overrightarrow{A}$ be the vector field $\Sigma a_i \frac{\partial}{\partial x^i}$. Let S be a smooth orientable surface in $\mathbb{R}^3$, with smooth boundary, and $\gamma : [0,1]^2 \to \mathbb{R}^3$ be a 2-cube which maps 1-1 and onto S (with $\partial\gamma$ mapping 1-1 and onto the boundary curve β). Classical Stokes' Theorem states the following:

$$\oint_\beta (\overrightarrow{A} \cdot \frac{d\beta}{dt})\, dt = \iint_S (curl\, \overrightarrow{A}) \cdot \overrightarrow{N}\, d\,\mathrm{area}_S.$$

Here $\overrightarrow{N}$ is a unit normal vector field for S and $d\,\mathrm{area}_S = ||\gamma_* \frac{\partial}{\partial u^1} \times \gamma_* \frac{\partial}{\partial u^2}||\, du^1 \wedge du^2$. Explain why this follows from our general Stokes' Theorem.

(HINT: write things out. You can do it all in a few lines if you use all the information you are given. You might want to look at Q5 and Q6 on worksheet 8 for help with the $d\,\mathrm{area}_S$ expression.)

There are more integration questions to be found on the extra problem page on the webpage for the course.