

# Math 25a Homework 8 Solutions

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## 1 Alison's problems

(1) We know that the set  $M_{2 \times 2}(\mathbb{R})$  of  $2 \times 2$  real matrices forms a vector space over  $\mathbb{R}$  (with the usual definitions of addition and scalar multiplications for matrices).

(a) Show that the subspace  $W$  consisting of *symmetric* matrices is a subspace of  $M_{2 \times 2}(\mathbb{R})$ . (Note:  $W = \{A \in M_{2 \times 2}(\mathbb{R}) : A = A^T\}$ .)

(b) Find a basis for  $W$ . What is the dimension of  $W$ ?

(c) Find a basis for  $M_{2 \times 2}(\mathbb{R})$  which contains a basis for  $W$ .

*Solution.* (a). Let  $w_1, w_2$  be any symmetric matrices in  $M_{2 \times 2}(\mathbb{R})$ . We need to show that  $w_1 + w_2$  and  $cw_1$  are symmetric. Write out

$$w_1 = \begin{pmatrix} x_1 & y_1 \\ y_1 & z_1 \end{pmatrix}, w_2 = \begin{pmatrix} x_2 & y_2 \\ y_2 & z_2 \end{pmatrix}.$$

Then

$$cw_1 = \begin{pmatrix} cx_1 & cy_1 \\ cy_1 & cz_1 \end{pmatrix}, w_1 + w_2 = \begin{pmatrix} x_1 + x_2 & y_1 + y_2 \\ y_1 + y_2 & z_1 + z_2 \end{pmatrix}$$

is also symmetric. So  $W$  is a subspace.

(b) We claim that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a basis for  $W$  Linear independence: for  $a_1, a_2, a_3 \in \mathbb{R}$ , the linear combination

$$a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$$

is only 0 if all its entries are 0, that is, if  $a_1 = a_2 = a_3 = 0$ .

Spanning: Any symmetric matrix  $\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$  can be represented as a linear combination by

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So  $W$  has a basis of three elements, and has dimension 3.

(c) Extend the basis from (b) to the list

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

By the linear dependence lemma, if these vectors are linearly independent, one of them must lie in the span of the previous vectors. It can't be one of the first three, because they are linearly independent. Neither can it be the last one, because  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is asymmetric and the span of the first three contains only symmetric matrices. Hence the four matrices are linearly independent. Because  $\dim M_{2 \times 2}(\mathbb{R}) = 4$ , they are a basis.  $\square$

(2) Given a finite dimensional vector space  $V$  over a field  $F$ . We aim to show that  $V^{**} \cong V$ . Recall the handy bracket notation we discussed in class (Monday 21st November). We discussed the map  $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow F$  with  $(f, v) \mapsto \langle f, v \rangle = f(v)$ .

(a) Given any nonzero vector  $v \in V$ , show that there is a  $f \in V^*$  such that  $f(v) = 0$ .

(b) Define a map  $\phi : V \rightarrow V^{**}$  by  $\phi(v) = \langle \cdot, v \rangle$ . (That is, for  $f \in V^*$ ,  $\phi(v)(f) = \langle f, v \rangle = f(v)$ .) Show that this map is injective (you might need part (a)). Then show the map is an isomorphism by counting dimensions (carefully).

Remark: Once you write out the (short) proof, it might seem that we haven't really done anything in part (b). In fact, the isomorphism we constructed is *canonical*. If we have time, we'll see precisely what this means in a future HW. For now, roughly, a canonical isomorphism means that it is independent of any choices. Contrast this with previous work. We (now) know that in fact  $V^* \cong V$ . This is because we constructed a basis for  $V^*$  from a basis of  $V$  — but note this isomorphism depended on the choices made.

*Solution.* (a) The set  $\{v\}$  is linearly independent: extend it to a basis  $\{v_1 = v, v_2, v_3, \dots, v_n\}$ . By the last problem set, there is a functional  $f^1 \in V^*$  such that  $f^1(v_i) = 1$  if  $i = 1$  and 0 otherwise. So  $f^1(v) = f^1(v_1) = 1 \neq 0$ .

(b) We note that our map  $\phi$  is linear because  $\langle f, v \rangle$  is linear in  $v$ . Now we show that its kernel is trivial. Consider any nonzero  $v$ . Then by (a), there exists an  $f$  such that  $\langle f, v \rangle \neq 0$ , so  $\phi(v)(f) \neq 0$ . This means that  $\phi(v)$  is not the zero map for any nonzero  $v$ , and  $\text{Ker } \phi$  must contain only the zero vector. Hence  $\phi$  is injective.

Now we count dimensions. From the last problem set we know that  $\dim V^* = \dim V$ , so also  $\dim V^{**} = \dim V^* = \dim V$ . Also,  $\dim V = \dim \text{Ker } \phi + \dim \text{Im } \phi = \dim \text{Im } \phi$  by injectivity. So  $\dim \text{Ker } \phi$  is a subspace of  $V$  with the same dimension as  $V$ : that must be all of  $V$ , so  $\phi$  is surjective as well as injective, hence an isomorphism.  $\square$

(3) Recall from class (Monday 21st November), that we defined the annihilator of a subset of a vector space. Given a vector space  $V$  and subset  $S$  (note:  $S$  does not have to be a subspace of  $V$ !), then the annihilator  $S^0 := \{f \in V^* : \langle f, s \rangle = 0, \forall s \in S\}$ .

(a) If  $W$  is an  $m$ -dimensional subspace of an  $n$ -dimensional vector space  $V$ , then show that  $W^0$  is an  $(n - m)$ -dimensional subspace of  $V^*$ .

(b) Show that if  $W$  is a subspace of a finite dimensional vector space  $V$ , then  $W^{00} = (W^0)^0 = W$ .

(c) Show that if  $S$  is any subset of a finite-dimensional vector space  $V$ , then  $S^{00}$  is the same as the subspace of  $V$  spanned by vectors in  $S$ .

(d) If  $S$  and  $T$  are subspaces of a finite-dimensional vector space  $V$  and if  $S \subset T$ , then  $T^0 \subset S^0$ .

(e) If  $W$  and  $U$  are subspaces of a finite-dimensional vector space  $V$ , then  $(W \cap U)^0 = W^0 + U^0$  and  $(W + U)^0 = W^0 \cap U^0$ .

*Solution.* (a) Let  $v_1, \dots, v_m$  be a basis of  $W$ , and extend it to a basis  $v_1, \dots, v_n$  of  $V$ . Consider the map  $\phi : V^* \rightarrow \mathbb{F}^m$  given by  $\phi(f) = (f(v_1), f(v_2), \dots, f(v_m))$ . The kernel of  $\phi$  is exactly  $W^0$ : for any  $f \in W^0$ , all the entries of  $\phi(f)$  are zero because  $v_1, v_2, \dots, v_m \in \text{Ker } f$ . Conversely, if  $\phi(f) = 0$ , for any  $v \in V$ , write  $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$ . Then  $f(v) = a_1f(v_1) + a_2f(v_2) + \dots + a_nf(v_n) = 0$ . So  $W^0 = \text{Ker } \phi$  is a subspace (we've shown before that the kernel of any linear map is a subspace.) We now need to check its dimension.

Our map  $\phi$  is surjective by problem set 9: given any  $(a_1, a_2, \dots, a_m) \in \mathbb{F}^m$  we can find  $f \in V^*$  such that  $f(v_i) = a_i$  for  $i = 1, \dots, m$ , and  $f(v_i)$  is whatever we want for  $i > m$ . Then this  $f$  satisfies  $\phi(f) = (a_1, a_2, \dots, a_m)$ : because our  $a$ 's could be anything, the map is surjective. So  $\dim \text{Im } \phi = \dim \mathbb{F}^m = m$ . Also  $n = \dim V = \dim \text{Ker } \phi + \dim \text{Im } \phi = \dim W^0 + m$ , so  $\dim W^0 = n - m$ .

(b) Counting dimensions again! First we prove that  $W \subset W^{00}$ . This is straight from the definition: for  $w \in W$  and any  $f \in W^0$ ,  $\langle w, f \rangle = \langle f, w \rangle = 0$ , so  $w \in W^{00}$  (note that we are identifying  $w$  with its canonical image, as given by problem 2, in  $V^{**}$ . This means that  $W \subset W^{00}$ . On the other hand, if  $\dim W = m$ ,  $\dim W^0 = n - m$ , and  $\dim W^{00} = \dim V^* - \dim W^0 = n - (n - m) = m = \dim W$ . So the two spaces have the same dimension, and one is contained in the other. They must be equal.

(c) We show that  $S^0 = (\text{Span } S)^0$ . The inclusion  $(\text{Span } S)^0 \subset S^0$  is clear, because if  $f \in (\text{Span } S)^0$ , for any  $v \in S$ ,  $v \in \text{Span } S$  also, so  $f(v) = 0$ , and  $f \in S^0$  as well. For the other one, let  $f \in S^0$ , and take any  $v \in \text{Span } S$ . By definition,  $v$  is a linear combination  $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$  of vectors  $v_1, \dots, v_n$  of  $S$ . By linearity,  $f(v) = a_1f(v_1) + a_2f(v_2) + \dots + a_nf(v_n) = 0$  because  $f$  annihilates each  $v_i \in S$ . So  $f \in (\text{Span } S)^0$  also,  $S^0 \subset (\text{Span } S)^0$ , and the two sets are equal.

Now we get to apply (b). Taking annihilators of  $S^0 = (\text{Span } S)^0$ ,  $S^{00} = (\text{Span } S)^{00} = \text{Span } S$ , which is what we wanted.

(d) Because  $S$  is a subset of  $T$ , anything in  $V^*$  that sends all elements of  $T$  to 0 also sends all elements of  $S$  to 0. So  $T^0 \subset S^0$ .

(e) We first prove the easier half:  $(W + U)^0 = W^0 \cap U^0$ . This is double inclusion. We have  $U, W \subset W + U$ , so by (d),  $(W + U)^0 \subset W^0, U^0$ , so  $(W + U)^0$  is also a subset of the intersection  $W^0 \cap U^0$ . That proves one inclusion,  $(W + U)^0 \subset W^0 \cap U^0$ . For the reverse inclusion, suppose that  $f \in W^0 \cap U^0$ . Then for all  $u \in U$ ,  $f(u) = 0$ , and for all  $w \in W$ ,  $f(w) = 0$ . By linearity, then, for all  $u + w \in U + W$ ,  $f(u + w) = f(u) + f(w) = 0$ , so  $f \in (W + U)^0$ . So  $W^0 \cap U^0 \subset (W + U)^0$ . By double inclusion,  $W^0 \cap U^0 = (W + U)^0$ .

It's possible, but a bit messy, to do the second part with bases or counting dimensions. Instead, we'll use a trick. In the result we've just proved,  $W^0 \cap U^0 = (W + U)^0$ , we replace  $W$  with  $W^0$  and  $U$  with  $U^0$ . This gives us  $(W^0 + U^0)^0 = W^{00} \cap U^{00} = W \cap U$ . Take annihilators of both sides:  $(W^0 + U^0)^{00} = (W \cap U)^0$ . By (b), this is just  $W^0 + U^0 = (W \cap U)^0$ .  $\square$

(4) (a) Read Chapter 4 of Axler

(b) Problem 5 page 73 of Axler. (I can think of two different proofs - can you? Note: just submit one for grading.)

*Solution.* (a) The solution is left as an exercise to the reader.

(b) By Axler 4.10, if  $\lambda$  is a root of  $f$ , so is  $\bar{\lambda}$ . This means that the non-real roots of  $f$  come in complex conjugate pairs. So if  $f$  had only non-real roots, it would have an even number of complex roots. By the Fundamental Theorem of Algebra, the number of complex roots of  $f$  equals the

degree of  $f$ , so  $f$  would have even degree. Conversely, if  $f$  has odd degree, it must instead have some real root.  $\square$