

# Math 25a Homework 12 Solutions

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## 1 Alison's problems

(1) Let  $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ .

(a) Use induction to prove that  $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$  for  $n \geq 1$ . Here  $M^n$  means  $M$  multiplied by itself  $n$  times and  $F_n$  is the  $n$ th Fibonacci number. That is,  $F_1 = F_2 = 1$  and  $F_{n+2} = F_{n+1} + F_n$  for  $n \geq 1$ .

(b) Find the eigenvalues of  $M$  and also an eigenvector for each eigenvalue. Express  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  as a linear combination of these eigenvectors.

(c) Use your answers to (a) and (b) to show that for  $n \geq 1$ ,

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].$$

*Solution.* (a) *Base Case:*  $n = 1$ .  $M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_3 \end{pmatrix}$ .

*Inductive Step:* Suppose that  $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$ . Then

$$M^{n+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = M M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+1} + F_{n+2} \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+3} \end{pmatrix}.$$

By induction,  $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$  for any positive integer  $n$ .

(b) Suppose that  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  is an eigenvector for  $M$  with eigenvalue  $\lambda$ . Equation components in the equation  $Mv = \lambda v$ , we get  $v_2 = \lambda v_1$ ,  $v_1 + v_2 = \lambda v_2$ . Substituting the first equation in the second, we find that  $(1 + \lambda - \lambda^2)v_1 = 0$ : because  $v$  is nonzero, we must have  $1 + \lambda - \lambda^2 = 0$ . By the quadratic formula, this implies that

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

Now if we take  $v_1 = 1$ , we have  $v_2 = \lambda$  from the first equation (and the second equation also checks).

So we have eigenvalues  $(1 + \sqrt{5})/2$ ,  $(1 - \sqrt{5})/2$  with corresponding eigenvectors  $\begin{pmatrix} 1 \\ (1 + \sqrt{5})/2 \end{pmatrix}$ ,

$\begin{pmatrix} 1 \\ (1 - \sqrt{5})/2 \end{pmatrix}$ .

To write  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  as a linear combination of the eigenvectors, let  $c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$ ,  $c_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}}$ . Using these as coefficients, we get

$$\frac{1+\sqrt{5}}{2\sqrt{5}} \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} + \frac{-1+\sqrt{5}}{2\sqrt{5}} \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} \\ \frac{3+\sqrt{5}-3+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

c) From a) and b), we have

$$\begin{aligned} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} &= M^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= M^{n-1} \left( c_1 \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \right) \\ &= \left( \frac{1+\sqrt{5}}{2} \right)^{n-1} c_1 \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} + \left( \frac{1-\sqrt{5}}{2} \right)^{n-1} c_2 \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}. \end{aligned}$$

Taking the first components of both sides, we get that

$$F_n = \left( \frac{1+\sqrt{5}}{2} \right)^{n-1} c_1 + \left( \frac{1-\sqrt{5}}{2} \right)^{n-1} c_2 = F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right].$$

□

(2) Let  $W$  be a subspace of a vector space  $V$ , and let  $T : V \rightarrow Z$  be a linear transformation. Let  $\pi : V \rightarrow V/W$  be the natural map from  $V$  to  $V/W$  defined by  $\pi(v) = [v]$ . (Recall that  $[v]$  is the equivalence class of  $v$  in  $V/W$  and  $v_1 \sim v_2$  if and only if  $v_1 - v_2 \in W$ .) Show that there exists a linear transformation  $\bar{T} : V/W \rightarrow Z$  such that  $T = \bar{T} \circ \pi$  if and only if  $W \subset \ker(T)$ .

*Solution. Proof of  $\Rightarrow$ :* Suppose  $T = \bar{T} \circ \pi$ . Let  $w \in W$  be arbitrary: we must show  $w \in \ker(T)$ , that is,  $Tw = 0$ . First,  $Tw = \bar{T}\pi w = \bar{T}[w]$ . Now,  $w \in W$ , so by definition of the quotient,  $[w] = [0]$ , and  $\bar{T}[w] = \bar{T}[0] = 0$ , as needed.

*Proof of  $\Leftarrow$ :* Suppose  $W \subset \ker(T)$ . If  $[v]$  is an arbitrary element of  $V/W$ , define  $\bar{T}[v] = Tv$ . We need to show that this is well-defined: suppose that  $[v_1] = [v_2]$ : then  $v_1 - v_2 \in W \subset \ker T$ . Hence  $0 = T(v_1 - v_2) = T(v_1) - T(v_2)$ , and  $T(v_1) = T(v_2)$ , which is what we need to have  $\bar{T}$  well-defined. Linearity of  $\bar{T}$  then follows automatically from the definition of addition of equivalence classes and the linearity of  $T$ :  $T[v + w] = T([v] + [w]) = T[v] + T[w]$  and  $T[cv] = T(c[v]) = cT[v]$ .

□

Problems (3), (4) and (5) will be dedicated to understanding the tensor product  $V \otimes W$  of two vector spaces  $V$  and  $W$ . (Note that these problems will be similar in style to the ones that you did on the product space  $\prod V_i$  of vector spaces  $V_i$  in HW 11.)

Given vector spaces  $V$  and  $W$  over a field  $F$  we define the tensor product  $V \otimes W$  to be a vector space over  $F$  together with a bilinear map<sup>1</sup>  $\mu : V \times W \rightarrow V \otimes W$  satisfying the following universal mapping property: Given any vector space  $U$  and bilinear map  $f : V \times W \rightarrow U$ , there is a unique linear transformation  $\hat{f} : V \otimes W \rightarrow U$  such that  $\hat{f} \circ \mu = f$ .

In problem (3) we'll construct  $V \otimes W$  and in problem (4) show it satisfies the universal mapping property for tensors. Finally in problem (5) we'll show that the UMP for tensors defines the tensor product uniquely up to isomorphism.

(3) Let  $X = \prod_{i \in I} F_i$ , where the indexing set  $I = V \times W$  and  $F_i = F$  (so  $F$  is viewed as a one-dimensional vector space over  $F$ ). Recall from HW 11 that this means

$$X = \{g : V \times W \rightarrow F \mid g((v, w)) = 0 \text{ for all but finitely many choices of } (v, w) \in V \times W\}.$$

(a) If we define  $\delta_{(v,w)} \in X$  by

$$\delta_{(v,w)}(v', w') = \begin{cases} 1 & \text{if } (v', w') = (v, w), \\ 0 & \text{otherwise,} \end{cases}$$

then show that  $\{\delta_{(v,w)} \mid (v, w) \in V \times W\}$  is a basis for  $X$  over  $F$ .

Let

$$Z = \text{span} \left( \{\delta_{(a_1 v_1 + a_2 v_2, w)} - a_1 \delta_{(v_1, w)} - a_2 \delta_{(v_2, w)}\} \cup \{\delta_{(v, b_1 w_1 + b_2 w_2)} - b_1 \delta_{(v, w_1)} - b_2 \delta_{(v, w_2)}\} \right)$$

for all  $a_1, a_2, b_1, b_2 \in F$ , all  $v, v_1, v_2 \in V$  and all  $w, w_1, w_2 \in W$ . Note that  $Z$  is the span of an infinite set<sup>2</sup> and note that  $Z$  is a subspace of  $X$ .

Define  $T := X/Z$  (a quotient of vector spaces) and  $\mu : V \times W \rightarrow T$  by  $\mu((v, w)) = [\delta_{(v,w)}]$  (the equivalence class of  $\delta_{(v,w)}$  in the quotient).

(b) Check that  $\mu$  is indeed bilinear.

*Solution.* a) Spanning: for any  $g \in X$ , we claim that  $g = \sum_{(v,w) \in V \times W} g(v, w) \delta_{(v,w)}$ : all but finitely many of the terms are 0, so this sum makes sense. We check this: for any  $(v_1, w_1) \in V \times W$ , in the  $\sum_{(v,w) \in V \times W} g(v, w) \delta_{(v,w)}(v_1, w_1) = g(v_1, w_1)$ , all terms are 0 except for  $g(v_1, w_1) \delta_{(v_1, w_1)}(v_1, w_1) = g(v_1, w_1)$ , so the functions agree.

Linear independence: suppose  $0 = \sum_{(v,w) \in V \times W} a_{v,w} \delta_{(v,w)}$  where only finitely many terms are nonzero (i.e. this is really a finite sum). Then for any  $(v, w) \in V \times W$ ,  $0 = \sum_{(v,w) \in V \times W} a_{v,w} \delta_{(v,w)}(v, w) = a_{v,w}$  (again all other terms drop out). That is, all the coefficients are 0, so the linear combination must be trivial. This shows linear independence.

b) It suffices to show that  $\mu(a_1 v_1 + a_2 v_2, w) = a_1 \mu(v_1, w) + a_2 \mu(v_2, w)$  and that  $\mu(v, a_1 w_1 + a_2 w_2) = a_1 \mu(v, w_1) + a_2 \mu(v, w_2)$ . For the first one, note that  $\mu(a_1 v_1 + a_2 v_2, w) = [\delta_{(a_1 v_1 + a_2 v_2, w)}]$ . By definition of  $Z$ ,  $\delta_{(a_1 v_1 + a_2 v_2, w)} - a_1 \delta_{(v_1, w)} - a_2 \delta_{(v_2, w)} \in Z$ , so  $[\delta_{(a_1 v_1 + a_2 v_2, w)}] = [a_1 \delta_{(v_1, w)} + a_2 \delta_{(v_2, w)}] = \mu(a_1 v_1, w) + \mu(a_2 v_2, w)$ . The same proof with  $v$  and  $w$  switched works for linearity in the second variable. □

<sup>1</sup>Recall that a bilinear map  $f : V \times W \rightarrow U$  is a map which is linear in each slot. That is  $f(a_1 v_1 + a_2 v_2, w) = a_1 f(v_1, w) + a_2 f(v_2, w)$  and  $f(v, b_1 w_1 + b_2 w_2) = b_1 f(v, w_1) + b_2 f(v, w_2)$  for all  $a_1, a_2, b_1, b_2 \in F$ , all  $v, v_1, v_2 \in V$  and all  $w, w_1, w_2 \in W$ .

<sup>2</sup>Recall that  $\text{span}(Y)$  where  $Y$  is infinite is defined to be the set of all *finite* linear combinations of elements in  $Y$ .

(4) Show that  $(T, \mu : V \times W \rightarrow T)$  satisfies the universal mapping property for tensor products. (Here  $T$  and  $\mu$  are defined as in problem (3).)

HINT 1: This means for any vector space  $U$  and any bilinear map  $f : V \times W \rightarrow U$  there exists a unique map  $\hat{f} : T \rightarrow U$  such that  $f = \hat{f} \circ \mu$ . So we have to construct such an  $\hat{f}$ , show it is linear, unique and show that  $f = \hat{f} \circ \mu$ .

HINT 2: To construct such an  $\hat{f}$ , we'll first define a linear map  $\bar{f} : X \rightarrow U$  by

$$\bar{f}(g) := \sum_{(v,w) \in V \times W} g((v,w))f((v,w)).$$

Note that this is a finite sum since  $g \in X$ . Now check that  $\bar{f}(Z) = \{0\}$ . Use problem (2) to conclude that we have a well-defined linear map  $\hat{f} : T \rightarrow U$  given by  $\hat{f}([g]) = \bar{f}(g)$ .

*Solution.* We first define  $\bar{f} : X \rightarrow U$  by

$$\bar{f}(g) := \sum_{(v,w) \in V \times W} g((v,w))f((v,w))$$

as in the hint. Linearity of  $\bar{f}$  follows directly from the definition of addition on  $X$ . We claim that  $Z \subset \text{Ker}(\bar{f})$ : it suffices to show that  $\bar{f}$  is 0 on a spanning set of  $Z$ . Note that for any  $v_1 \in V, w_1 \in W$ ,  $\bar{f}(\delta_{v_1, w_1}) = \sum_{(v,w) \in V \times W} \delta(v_1, w_1) f((v,w)) = f(v, w)$ . So for any  $v_1, v_2 \in V, w \in W$  and  $a_1, a_2 \in F$ ,  $\bar{f}(a_1\delta(v_1, w) + a_2\delta(v_2, w) - \delta_{a_1v_1 + a_2v_2, w}) = a_1f(v_1, w) + a_2f(v_2, w) - f(a_1v_1 + a_2v_2, w) = 0$  by linearity of  $f$  in  $v$ . Doing the same thing in the second variable,  $\bar{f}(a_1\delta(v, w_1) + a_2\delta(v, w_2) - \delta_{v, a_1w_1 + a_2w_2}) = 0$ . The space  $Z$  is defined as the spanning set of the vectors of the above form, so  $Z \subset \text{ker } \bar{f}$ .

We can now apply problem 2 to show that there is a well-defined map  $\hat{f} : T \rightarrow U$  given by  $\hat{f}([g]) = \bar{f}(g)$ . Then for any  $(v, w) \in V \times W$ ,  $(\hat{f} \circ \mu)(v, w) = \hat{f}([\delta_{v,w}]) = \bar{f}(\delta_{v,w})$ . But we showed in the previous paragraph that  $\bar{f}(\delta_{v,w}) = f(v, w)$ . Hence  $(\hat{f} \circ \mu)(v, w) = f(v, w)$ : this is true for any  $(v, w)$ , so  $\hat{f} \circ \mu = f$ .

Now we need uniqueness. Suppose we had another such map  $\tilde{f} : T \rightarrow U$  satisfying the above properties. Then for any  $(v, w) \in V \times W$ ,  $\tilde{f}([\delta_{v,w}]) = (\tilde{f} \circ \mu)(v, w) = f(v, w) = \hat{f}([\delta_{v,w}])$ . So  $\tilde{f}$  and  $\hat{f}$  take the same values at any vector of the form  $[\delta_{v,w}]$ . Because the set of all  $\delta_{v,w}$  is a basis for  $X$ , after quotienting the set of all  $[\delta_{v,w}]$  still spans  $T = X/Z$ . Hence  $\tilde{f}$  and  $\hat{f}$  agree on a spanning set, so they must be the same map. That is,  $\hat{f}$  is unique. □

(5) Suppose that  $(T', \mu' : V \times W \rightarrow T')$  is another pair satisfying the universal mapping property of a tensor product. Show that  $T \cong T'$ . (HINT: This is exactly the same style of argument as for coproducts and products that you saw in HW 11.)

*Solution.* We do the same back-and-forth with universal mapping properties as in homework 11. The map  $\mu$  is bilinear from  $V \times W$  to  $T$ , so we can use the universal mapping property of  $T'$  to write  $\mu = \hat{\mu} \circ \mu'$  for some linear map  $\hat{\mu} : T' \rightarrow T$ . In the other direction, we can do the same argument with  $T$  and  $T'$  switched, to write  $\mu' = \hat{\mu}' \circ \mu$  for some  $\hat{\mu}' : T \rightarrow T'$ . We now claim that  $\hat{\mu}, \hat{\mu}'$  are inverse maps. Composing, we see that  $\mu = \hat{\mu} \circ \hat{\mu}' \circ \mu$ . However, also  $\mu = I_T \circ \mu$ : by the uniqueness condition in the universal mapping property, we must have  $\hat{\mu} \circ \hat{\mu}' = I_T$ . Similarly,  $\mu' = \hat{\mu}' \circ \hat{\mu} \circ \mu' = I_{T'} \circ \mu'$ , so the two maps are inverses, and provide an isomorphism between  $T$  and  $T'$ . □

FINAL REMARKS: The tensor product of  $V$  and  $W$  is usually denoted  $V \otimes W$  and the image  $\mu(\delta_{(v,w)}) = [\delta_{(v,w)}]$  is generally written as  $v \otimes w$ . Make sure that you understand that in this notation  $(a_1v_1 + a_2v_2) \otimes w = a_1(v_1 \otimes w) + a_2(v_2 \otimes w)$  and  $v \otimes (b_1w_1 + b_2w_2) = b_1(v \otimes w_1) + b_2(v \otimes w_2)$ .