Math 25a Homework 12

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1 Ivan's problems

(1) (a) Problem 24 on page 125 of Axler. (Hint: use your answer to question 10.)

Solution. Find an orthonormal basis and use it to construct $q(x) = -3/2 + 15x - 15x^2$. □

(b) Problem 27 on page 125 of Axler.

Solution. Fix $(z_1, \ldots, z_n) \in F^n$. Then for every $(w_1, \ldots, w_n) \in F^n$, we have

$$\langle (w_1, \ldots, w_n), T^*(z_1, \ldots, z_n)\rangle = \langle T(w_1, \ldots, w_n), (z_1, \ldots, z_n)\rangle$$

$$= \langle (0, w_1 \ldots, w_{n-1}), (z_1, \ldots, z_n)\rangle = w_1\bar{z_2} + \cdots + w_{n-1}\bar{z_n} = \langle (w_1, \ldots, w_n), (z_2, \ldots, z_n, 0)\rangle$$

Thus $T^*(z_1, \ldots, z_n) = (z_2, \ldots, z_n, 0)$. □

(2) (a) Problem 28 on page 125 of Axler.

Solution. We have

$$\begin{aligned} \lambda \text{ is not an eigenvalue of } T &\Leftrightarrow T - \lambda I \text{ is invertible} \\ &\Leftrightarrow S(T - \lambda I) = (T - \lambda I)S = I \\ &\qquad \text{for some } S \in L(V) \\ &\Leftrightarrow (T - \lambda I)^* S^* = S^*(T - \lambda I)^* = I \\ &\qquad \text{for some } S \in L(V) \\ &\Leftrightarrow (T - \lambda I)^* \text{ is invertible} \\ &\Leftrightarrow T^* - \bar{\lambda} I \text{ is invertible} \\ &\Leftrightarrow \bar{\lambda} \text{ is not an eigenvalue of } T^* \end{aligned}$$

Thus λ is an eignevalue of T iff $\bar{\lambda}$ is an eigenvalue of T^*. □

(b) Problem 29 on page 125 of Axler.

Solution. First suppose that U is invariant under T. To prove that $U^\perp$ is invariant under T^*, let $v \in U^\perp$. We need to show that $T^*v \in U^\perp$. But $\langle u, T^v \rangle = \langle Tu, v \rangle = 0$ for every $u \in U$ (beacuse if $u \in U$ then $Tu \in U$ and hence Tu is orthogonal to v, an element in $U^\perp$). Thus $T^*v \in U^\perp$ and hence $U^\perp$ is invariant under T^* as desired.

To prove the other direction now suppose that $U^\perp$ is invariant under T^*. Then by the first direction we know that $(U \perp) \perp$ is invariant under $(T^*)^*$ is the same as U invariant under T. □

(3) (a) Problem 30 on page 125 of Axler.

Solution. For (a): T is injective $\Leftrightarrow nullT = \{0\} \Leftrightarrow (rangleT^*)^\perp = \{0\} \Leftrightarrow rangeT^* = W \Leftrightarrow$ T^* is surjective. The second claim follows replacing T with T^*. □

(b) Problem 31 on page 125 of Axler.

Solution. Let $T \in L(V, W)$ then $\dim nullT^* = \dim (rangeT)^\perp = \dim W - \dim rangeT = \dim nullT + \dim W - \dim V$, with the first equality from 6.46a and the second from exercise 15 and the third from 3.4. This proves the first equality.

For the second not that $\dim rangeT^* = \dim W - \dim nullT^* = \dim V - \dim nullT = \dim rangeT$ where the first and third equalities come from 3.4 and the second from the first part of the exercise. □

(4) (a) Problem 2 on page 158 Axler.

Solution. □

(b) Problem 4 on page 158 Axler.

Solution. Consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Clearly these provide a counter example. □

(5) (a) Problem 9 on page 159 Axler.

Solution. Suppose V is a complex inner product space and $T \in L(V)$ is normal. If T is self-adjoint then by 7.1 all its eigenvalues are real. Conversely, suppose that all the enigenvalues of T are real. By the complex spectral theorem there is an orthonormal basis e_i of V consisting of eigenvectors of T. thus there exist real numbers λ_i such that $Te_i = \lambda_i$. The matrix of T wrt the basis is diagonal with real values on the diagonal, hence equal to its conjugate transpose. Thus $T = T^*$, or T is self-adjoint. □

(b) Problem 10 on page 159 Axler.

Solution. By the complex spectral theorem there is an o.n. basis e_i of V of eigenvectors of T with eigenvalues λ_i. Thus $Te_i = \lambda e_i$. Applying T repeatedly to both sides gives $T^9 e_i = \lambda_i^9 e_i$ and same with exponent 8. Thus $\lambda_i^9 = \lambda_i^8$ so $\lambda_i = 0$ or 1. Thus real eigenvalues implies T self-adjoint and we know that for all i $\lambda^2 = \lambda$. So $T^e e_i = \lambda^2 e_i = \lambda e_i = Te_i$. Since T and T^2 agree on a basis, they extend to equal operators on the space. □