Math 25a Homework 6 Solutions

Ivan Corwin and Alison Miller.

1 Alison's problems

(1) Problem 6 on page 78 of Rudin.

Solution.

(a) $a_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n}+\sqrt{n+1}} = \frac{2}{\sqrt{n}+\sqrt{n+1}} > \frac{2}{\sqrt{n}}$. So, $\sum a_n$ diverges by comparison to (that is, Rudin Thm. 3.25) $\sum \frac{1}{\sqrt{n}}$ (see also, Rudin Thm. 3.28).

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{2}{n(\sqrt{n}+\sqrt{n+1})} < \frac{2}{n^{\frac{3}{2}}}$, so $\sum a_n$ converges by comparison to $\sum 2n^{-3/2}$.

(c) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n} - 1 = 0$, so $\sum a_n$ converges by the root test.

(d) For $|z| \leq 1$, $|1+z^n| \leq 2$ for all n by the triangle inequality. Hence $\left|\frac{1}{1+z^n}\right| \geq \frac{1}{2}$ for all n, so a_n does not go to 0, and $\sum a_n$ must diverge. For $|z| > 1$, $|a_n| = \frac{1}{|1+z^n|} \leq \frac{1}{|z|^n - 1} \leq \frac{1}{|z|-1}\frac{1}{|z|^{n-1}}$, so $\sum a_n$ converges by comparison to $\sum |z|^{1-n}$. □

(2) Problem 7 on page 78 of Rudin.

Solution. $(\Sigma a_n)(\Sigma 1/n^2) \geq (\Sigma\sqrt{a_n}/n)^2$ by Cauchy, so the partial sums are bounded since the two sequences on the left both converge. The sums are also monotonic, so they converge. Alternatively, you can use the AM-GM inequality (the thing we used on Ivan's problem 1 on Problem set 5) in the form $\sqrt{a_n}/n \leq (a_n + 1/n^2)/2$ to bound the partial sums. □

Note: A number of people tried to use the root or ratio tests on this problem (as well as some others on the homework where they weren't helpful). The problem with this approach is that when the lim sup of roots/ratios is 1, the test tells you nothing: and that is where the interesting borderline cases lie. (Think about what this function does to p-series, where the root and ratio tests both give 1.)

(3) Problem 8 on page 79 of Rudin.

Without loss of generality, we can assume that $\{b_n\}$ is monotone decreasing (otherwise, replace $\{b_n\}$ with the sequence $\{-b_n\}$. Then $\{b_n\}$ is monotone and bounded, so it has some limit B. Then the sequence $\{b_n - B\}$ is monotone decreasing and converges to 0, so by Rudin 3.42, $\sum_{n=1}^{\infty} a_n(b_n - B)$ converges. Also, because B is constant, $\sum_{n=1}^{\infty} a_n B$ converges.

Hence $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n(b_n - B) + \sum_{n=1}^{\infty} a_n B$ converges.

(*Note:* You can also prove this directly by imitating the proof of Rudin 3.42.)

(4) Problem 10 on page 79 of Rudin.

Solution. Note that $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$ (for there are non-zero, and so absolute value at least 1, a_n arbitrarily far in the sequence). So, $R = \frac{1}{\alpha} \leq 1$. □