Math 25a Homework 9 Solutions

Ivan Corwin and Alison Miller.

1 Ivan's problems

(1) (a) Let V be a vector space and let U and W be subspaces of V such that $U \oplus W = V$. Define a map $\phi : U \to V/W$ such that $\phi(u) = [u]$. Show that ϕ is an isomorphism. (That is show ϕ is linear, injective and surjective). Hence $U \cong V/W$.

(Hint: Recall that V/W is the quotient space of V with W, with $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$ and the equivalence class of v is denoted by $[v]$ — see HW 8).)

(b) If V is a finite dimensional vector space, W a subspace of V with $\dim V = n$ and $\dim W = m$, then using part (a) or otherwise, prove that $\dim V/W = n - m$.

Solution. (a) First observe that ϕ is well-defined by HW 8. Linearity clear follows from the equivalence classes. Injectivity follows since if $\phi(a) = \phi(b)$ then $[a] = [b]$ and hence $a - b \in W$ but also $a - b \in U$ so $a - b = 0$ (the only element of $U \cap W$). Surjectivity follows since for any element $[u] \in V/W$ we can find $u \in U$ which maps to it.

(b) We know that $dimU + dimW = dimV$ but since U and V/W are isomorphic, we have $dimU = dimV/W$. So replacing $dimW$ with m and $dimV$ with n we get $dimV/W = n - m$. □

(2) Explain how and why a linear transformation $T : V \to W$ induces an isomorphism between $V/ker(T)$ and $im(T)$.

Solution. A similar argument to the previous question suffices to explain this. □

(3) Problem 13 on page 60 of Axler.

Solution. First suppose that there exists $T \in L(V,W)$ such that $nullT = U$. Then $dimU = dimnullT = dimV - dimrangeT \geq dimV - dimW$, where the second equality comes from 3.4.

To prove the other direction, now suppose that $dimU \geq dimV - dimW$. Let $(u_1, \ldots, u_m)$ be a basis for U. Extend to a basis $(u_1, \ldots, u_m, v_1 \ldots, v_n)$ of V. Let $(w_1, \ldots, w_p)$ be a basis for W. For $a_1, \ldots, a_m, b_1, \ldots, b_n \in F$ define $T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n)$ as $b_1w_1 + \cdots + b_nw_m$. Because $dimW \geq dimV - dimU$, we have $p \geq n$ and so w_n on the right side of the equation above makes sense. Clearly $T \in L(V,W)$ and $nullT = U$. □

(4) Problem 14 on page 60 of Axler. (Hint: you might need problem 3 on page 59 of Axler.)

Solution. First suppose that T is injective. Define $S' : rangeT \leftarrow V$ by $S'(Tv) = v$. Because T is injective, each element of $rangeT$ can be represented in the form Tv in only one way, so S' is well defined. As can be easily checked, S' is a linear map on $rangeT$. Furthermore S' can be extended to a linear map $S \in L(W,V)$. If $v \in V$, then $(ST)v = S(Tv) = S'(Tv) = v$ thus ST is the identity map on V.

In the other direction, is such an S exists then for $u, v \in W$, if $Tu = Tv$ then $STu = STv$ implies $u = v$. □

(5) Problem 16 on page 60 of Axler.

Solution. Define a linear map $T' : nullST \leftarrow V$ by $T'u = Tu$. If $u \in nullST$ then $S(Tu) = 0$ which means that $Tu \in nullS$. In other words, $rangeT' \subset nullS$. Now

$$dimnullST = dimnullT' + dimrangeT' \leq dimnullT' + dimnullS \leq dimnullT + dimnullS,$$

where the first equality follows from 3.4, the first inequality from $rangeT' \subset nullS$ and the second from $nullT' \subset nullT$. □

(6) Problem 26 on page 61 of Axler.

Solution. Define $T \in L(F^n)$ by

$$T(x_1, \ldots, x_n) = (\sum_{k=1}^{n} a_{1,k}x_k, \ldots, \sum_{k=1}^{n} a_{n,k}x_k).$$

Then (a) above is the assertion that T is injective and (b) above is the assertion that T is surjective. By 3.21 these two assertions are equivalent. □