Math 25a Homework 7 Solutions

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1 Alison's problems

(1) Problem 5 on page 19 of Axler.

Solution. (a) $W_1 = \{(x_1, x_2, x_3) \in F^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$ Yes. We have two things to check here.

Closure under addition: Suppose that $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are elements of W_1. The vector $v + w$ is given by $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$, and

$$(v_1 + w_1) + 2(v_2 + w_2) + 3(v_3 + w_3) = (v_1 + 2v_2 + 3v_3) + (w_1 + 2w_2 + 3w_3) = 0 + 0 = 0.$$

This means that $v + w \in W_1$ also.

Closure under scalar multiplication: Suppose $v = (v_1, v_2, v_3) \in W_1$, and let $c \in F$ be an arbitrary scalar. The vector $5v$ is equal to $(5v_1, 5v_2, 5v_3)$, and

$$(cv_1) + 2(cv_2) + 3(cv_3) = 5(c_1 + 2r_2 + 3c_3) = 5(0) = 0$$

which means that $cv \in W_1$ also.

(b) $W_2 = \{(x_1, x_2, x_3) \in F^3 \mid x_1 + 2x_2 + 3x_3 = 4\}$.

This is not a subspace, for if it were, it would have to contain the zero element $(0, 0, 0)$ of F^3. But $0 + 2(0) + 3(0) = 0 \neq 4$, so $0 \notin W_2$, and W_2 is not a subspace.

(c) $W_3 = \{(x_1, x_2, x_3) \in F^3 \mid x_1x_2x_3 = 0\}$.

This is not a subspace, because it does not satisfy the additivity criterion. For a counterexample, consider $v = (1, 0, 0)$ and $w = (0, 1, 1)$. Then v and w are both elements of W_3, but $vw = (1, 1, 1)$ is not.

(d) $W_4 = \{(x_1, x_2, x_3) \in F^3 \mid x_1 = 5x_3\}$

Yes, this is a subspace: to prove it, we have to check our two closure properties.

Closure under addition: Suppose that $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are elements of W_4, so $v_1 = 5v_3$, $w_1 = 5w_3$. We need to check that $v+w = (v_1 +w_1, v_2 +w_2, v_3 +w_3)$ is also an element of W_4. In fact, $v_1 + w_1 = 5v_3 + 5w_3 = 5(v_3 + w_3)$, so $v + w \in W_4$, as we wanted.

Closure under scalar multiplication: Let $v = (v_1, v_2, v_3)$ be an element of W_1, and let c be any scalar. The vector $cv = (cv_1, cv_2, cv_3)$ is also an element of W_1 because $cv_1 = c(5v_3) = 5(cv_3)$.

□

(2) Problem 6 on page 19 of Axler.

Solution. Let $U = \mathbb{Z} \times \mathbb{Z} = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$. This is closed under addition and additive inverses because $\mathbb{Z}$ is closed under addition and additive inverses. However, it is not closed under scalar multiplication because $(1, 1) \in U$, $1/2 \in \mathbb{R}$, but $1/2(1, 1) = (1/2, 1/2) \notin U$. □

(3) Problem 9 on page 19 of Axler.

Let the subspaces be $V_1, V_2 \subset V$

Proof of $\Rightarrow$*:* We argue by contradiction. Suppose that $V_1 \not\subset V_2$, and $V_2 \not\subset V_1$. Then we can find $v_1 \in V_1$ such that $V_1 \notin V_2$, and also $v_2 \in V_2$ such that $v_2 \notin V_1$. Then v_1, v_2 are both elements of $V_1 \cup V_2$: because this is a vector space, $v_1 + v_2 \in V_1 \cup V_2$. Without loss of generality, say $v_1 + v_2 \in V_1$. Then $v_1 + v_2 \in V_1$, $-v_1 \in V_1$ (by closure under scalar multiplication). Because V_1 is closed under addition, $(v_1 + v_2) + (-v_1) = v_2$ must also lie in V_1. However we chose v_2 to not be an element of V_1. This is a contradiction, so our assumption that $V_1 \not\subset V_2$, $V_2 \not\subset V_1$ must be incorrect. Hence one of V_1, V_2 must actually be a subset of the other.

Proof of $\Leftarrow$*:* Without loss of generality, assume that $V_1 \subset V_2$. Then $V_1 \cup V_2 = V_2$, which is a subspace of V by definition.

2 Ivan's problems

(1) Problem 13 on page 19 of Axler.

Solution. Counterexamples: Consider V any nonzero vector space and $U_1 = \{0\}$, $U_2 = V$, and $W = V$, then $U_1 + W = U_2 + W = V$ but $U_1 \neq U_2$. □

(2) Problem 15 on page 19 of Axler.

Solution. Counterexample: Let $V = F^2$ and $U_1 = \{(x, 0) : x \in F\}$ and $U_2 = \{(0, y) : y \in F\}$ and $W = \{(z, z) : z \in F\}$. Then $F^2 = U_1 \oplus W = U_2 \oplus W$ but $U_1 \neq U_2$. □

(3) Problem 1 on page 35 of Axler.

Solution. Suppose $(v_1, \ldots, v_n)$ spans V. Let $v \in V$. To show that $v \in span(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$, we need to find $a_1, \ldots, a_n \in F$ such that $v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n$. Rearranging terms we see that we need to find $a_1, \ldots, a_n \in F$ such that $v = a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + \cdots + (a_n - a_{n-1})v_n$. Since $(v_1, \ldots, v_n)$ spans V, there exist $b_1, \ldots, b_n \in F$ such that $v = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_n v_n$. Comparing these two equations we see that if we choose $a_1 = b_1$ and $a_2 = b_2 + a_1$ and $a_3 = b_3 + a_2$ and so on, we get our desired results. □

(4) Problem 2 on page 35 of Axler.

Solution. Suppose $(v_1, \ldots, v_n)$ is linearly independent in V. To prove that the list displayed above is linearly independnent, suppose $a_1, \ldots, a_n \in F$ are such that $a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n = 0$. Rearranging terms, we get $a_1 + v_1 + (a_2 - a_1)v_2 + \cdots + (a_n - a_{n-1})v_n = 0$ and since that v_i are linearly independent, $a_1 = 0$, $a_2 - a_1 = 0$, ... , $a_n - a_{n-1} = 0$ and by the first equation this implies that $a_i = 0$ for all i and hence $(v_1 - v_2, \ldots v_{n-1} - v_n, v_n)$ is linearly independent. □