Math 25a Homework 8 Solutions

Ivan Corwin and Alison Milller.

1 Alison's problems

(1) Problem 1 on page 59 of Axler

Solution. Suppose $\dim V = 1$ and $T \in \mathcal{L}(V, V)$. Let u be any nonzero vector in V: then u spans V, so any vector in V is a scalar multiple of u. In particular, we can write $Tu = au$ for some $a \in F$.

Now let $v \in V$ be arbitrary: we can also write v as a scalar multiple of u, that is, $v = bu$ for $b \in F$. Now we apply T, and use linearity: $Tv = T(bu) = b(Tu) = b(au) = a(bu) = av$. Because v was arbitrary, $Tv = av$ for all $v \in V$. □

Note: A common mistake on this problem was to write Tv as av where a was some expression written in terms of things that depended upon the choice of v. Part of the point of the problem is to show that not only is Tv a constant multiple of v, it is the same constant multiple regardless of your choice of v. For this, linearity is essential.

(2) (a) Problem 9 on page 59 of Axler.
(b) Let $\mathcal{P}_n$ be the set of all real polynomials of degree less than n. Consider the linear transformations $S : \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $S(f) = \int_0^x f(t)dt$ and $T : \mathcal{P}_{n+1} \to \mathcal{P}_n$ such that $T(g) = g'$.

(i) Compute the matrix A of S with respect to the standard bases for $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$.

(ii) Compute the matrix B of T with respect to the standard bases for $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$.

(iii) Compute the matrix products AB and BA. What do you observe — any suggestions as to why this happens?

Solution. (a) Suppose we have such a T. Then the null space $\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \mid x_1 = 5x_2 \text{ and } x_3 = 7x_5\}$ has as a basis the two vectors $(1, 5, 0, 0)$ and $(0, 0, 1, 7)$. Hence $\operatorname{null} T$ has dimension 2. From Theorem 3.4 we see that $\dim \operatorname{range} T = \dim F^4 - \dim \operatorname{null} T = 4 - 2 = 2$, so $\operatorname{range} T$ is a two-dimensional subspace of $\mathbb{R}^2$. So $\operatorname{range} T$ must equal $\mathbb{R}^2$ itself, which is to say T is surjective.

(b) (i) Our kth basis vector is x^k, which, when integrated becomes $x^{k+1}/(k+1)$. Putting this all together, we get the $n+1$ by n matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1/2 & 0 & \dots & 0 \\ 0 & 0 & 1/3 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/k \end{pmatrix}$$

(ii) This time, we differentiate our basis vector x^k and get kx^{k-1}, so we end up with an n by $n+1$ matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & k-1 \end{pmatrix}.$$

(iii) When we multiply the matrices, we end up with AB being the $n+1$ by $n+1$ matrix

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and BA is an n by n matrix:

$$BA = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Note that BA is the identity: when you integrate and then differentiate, you get the original polynomial back. However, AB is not: when you differentiate and then integrate, you lose the constant terms. So the two matrices are different, and not only in their size. $\square$

(3) Let V be a finite dimensional vector space over a field F and let $v_1, \dots, v_n$ be a basis for V.

(a) If $a_, \dots, a_n \in F$, then prove that there is a **unique** $f \in V^*$ such that $f(v_1) = a_1$, $f(v_2) = a_2$, $\dots$, $f(v_n) = a_n$. (Recall V^* is the dual space of V.)

(b) Using part (a), define $f^i : V \to F$ by

$$f^i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Prove that $\{f^1, f^2, \dots, f^n\}$ is a basis for V^*.

Solution. (a) We can define such an f as follows: if $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, $f(v) = a_1c_1 + a_2c_2 + \dots a_nc_n$. This is well defined because $v_1, \dots, v_n$ is a basis, and it is easily checked to be linear. Additionally, for $v = v_i$, all terms but the ith are 0, so we are left with $f(v) = a_i$ as needed.

Now to show uniqueness: let f' be some other linear functional such that $f'(v_1) = a_1$, $f'(v_2) = a_2$, $\dots$, $f'(v_n) = a_n$. Then for any $v \in V$, we can write $v = c_1v_1 + c_2v_2 + \cdots c_nv_n$. Applying the linearity of f', we see that

$$f'(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1f'(v_1) + c_2f'(v_2) + \cdots c_nf'(v_n) = a_1c_1 + a_2c_2 + \dots a_nc_n = f(v).$$

So f' takes on the same values as f, which means that they are the same function. That is, f is the unique function with the properties above.

(b) First we show that $f^1, f^2, \dots, f^n$ are linearly independent. Suppose not. Then there is some linear dependence relation of the form $c_1 f^1 + c_2 f^2 + \dots c_n f^n = 0$, where $c_i \neq 0$ for some i. Then we can plug in the vector v_i and use the definition of linear combinations of functionals:

$$0 = 0(v_i) = (c_1 f^1 + c_2 f^2 + \dots c_n f^n)(v_i) = c_1 f^1(v_i) + c_2 f^2(v_i) + \cdots + c_n f^n(v_i) = c_i$$

because $f^j(v_i)$ is 0 except when $j = i$, where it is 1. However, we chose i such that c_i was nonzero, so this is a contradiction. Hence $f^1, f^2, \dots, f^n$ are linearly independent.

Now we show that $f^1, f^2, \dots, f^n$ span. Let f be an arbitrary element of V^*, and define $a_i = f(v_i)$ for $i = 1, 2, \dots, n$. Let $f' = a_1 f^1 + a_2 f^2 + \cdots + a_n f^n$. We claim that $f' = f$. By the argument above, for each $i = 1, 2, \dots, n$, $f'(v_i) = a_i = f(v_i)$. By the uniqueness property proved in part (a), this means that f must equal f', so $f \in \operatorname{Span}(f^1, f^2, \dots, f^n)$. Since f was arbitrary, this span must be the whole of V. □

(4) Problem 11 on page 60 of Axler. (Warning: you can't use Theorem 3.4 of Axler.)

Solution. Suppose T is a linear map from V into another vector space such that $\operatorname{null} T$ and $\operatorname{range} T$ are both finite dimensional. Then let $u_1, u_2, \dots, u_m$ be a basis of $\operatorname{null} T$ and let $w_1, w_2, \dots, w_n$ be a basis of $\operatorname{range} T$. Additonally, each $w_j \in \operatorname{range} T$, so for each one we can pick a $v_j \in V$ such that $Tv_j = w_j$.

Now we claim that $(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n)$ spans V. Let v be an arbitrary element of V. Consider $Tv \in \operatorname{range} T$. It can be expressed in terms of our basis for $\operatorname{range} T$: $Tv = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$ for some $c_1, c_2, \dots, c_n \in F$. By definition of $v_1, v_2, \dots, v_n$, and by linearity, we can rewrite this as

$$Tv = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n = c_1 T v_1 + c_2 T v_2 + \cdots + c_n T v_n = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n).$$

Subtracting off the right hand side, we see that $v - c_1 v_1 - c_2 v_2 - \cdots - c_n v_n \in \operatorname{null} T$, so we can express it in terms of our basis:

$$v - c_1 v_1 - c_2 v_2 - \cdots - c_n v_n = b_1 u_1 + b_2 u_2 + \cdots + b_n u_m$$

for some $b_1, b_2, \dots, b_n \in F$. Rearranging again,

$$v = b_1 u_1 + b_2 u_2 + \cdots + b_n u_m + c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

We can do this for any $v \in V$, so the set $u_1, \dots u_m, v_1, \dots, v_n$ spans V. Because V is spanned by a finite set, it is in fact finite dimensional. □

(5) Problem 17 on page 60 of Axler.

Solution. It is fairly straightforward and not particularly messy to do this problem by direct calculation with matrices, which is what most of you did. I will give an alternative solution involving linear transformations.

In order for the sizes to be right, B and C must have the same size, say n-by-p, and A must be the right size to multiply it by B or C: say A is m-by-n. Then consider linear transformations corresponding to multiplying by the matrices A, B, and C. That is, take $R \in \mathcal{L}(F^n, F^m)$, $S, T \in \mathcal{L}(F^p, F^n)$ so that they have for their matrices $\mathcal{M}(R) = A$, $\mathcal{M}(S) = B$, and $\mathcal{M}(T) = C$ (Axler 3.19). Because matrix multiplication and addition correspond to addition and multiplication of linear tranformations, $A(B+C) = \mathcal{M}(R(S+T))$, and $AB + AC = \mathcal{M}(RS + RT)$. So it suffices to show that $RS + RT = R(S+T)$. Indeed, for any $v \in F^p$ we can apply the definitions of addition and multiplication of linear maps to get $(RS + RT)(v) = (RS)v + (RT)v = R(Sv) + R(Tv)$ and $(R(S+T))(v) = R((S+T)(v)) = R(S(v) + T(v)) = R(S(v)) + R(T(v))$ (we use linearity of R for the last step). So $RS + RT$ and $R(S+T)$ are the same linear transformation, and their matrices, $AB + AC$ and $A(B+C)$, are also equal. □

(6) Problem 22 on page 61 of Axler.

Solution. Proof of $\Rightarrow$*:* Suppose ST is invertible: by Axler 3.21, ST is both injective and surjective. We first do T: let v be an arbitrary element of $\operatorname{null} T$. Then $Tv = 0$, so $STv = 0$ also, and $v \in ST$. But ST is injective, so $v = 0$. Hence $\operatorname{null} T = 0$, and T is injective. By Axler 3.21, T is invertible. For S, let v be any vector in V: because ST is surjective, we can find $w \in V$ such that $v = STw$. Then $v = S(Tw)$, so $v \in \operatorname{range} S$ also. This means that S is surjective, so it is invertible by Axler 3.21.

Proof of $\Leftarrow$*:* This can also be done with Axler 3.21, but we will instead construct an inverse explicitly. Suppose S, T are invertible, with inverses S^{-1}, T^{-1}. We claim that $T^{-1}S^{-1}$ is an inverse for ST. To check, note that $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SS^{-1} = I$. Likewise, $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}T = I$. So $T^{-1}S^{-1}$ works as our inverse. □