

Math 25a/55a – Honors Advanced Calculus and Linear Algebra
Problem Set A, due Monday September 22.

1. Given a set X , let $S^n(X)$ be the set of all subsets of X with exactly n elements. If A and B are in $S^n(X)$, define

$$d(A, B) = \text{the number of elements of } A \text{ which are not in } B.$$

Prove that d is a metric on $S^n(X)$.

2. Suppose a function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfies: $f(0) = 0$ and $f(x) < f(y)$ for $x < y$ and $f(x + y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}_{\geq 0}$. If (X, d) is a metric space, show that

$$d'(x, y) = f(d(x, y))$$

is also a metric.

Do Rudin, p. 44 exercise 11.

3. Given a metric space (X, d) , and a point x_0 in X , define a function f by

$$f(x) := d(x, x_0).$$

Show that f is continuous. (Hint: first show that $|d(x, y) - d(x, z)| \leq d(y, z)$ for any x, y, z).

4. Recall the three metrics we discussed on \mathbb{R}^n : if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we have

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|,$$

$$d_2(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}, \text{ and}$$

$$d_\infty(x, y) = \max_i |x_i - y_i|.$$

- Check that $d_1(x, y) \geq d_2(x, y) \geq d_\infty(x, y)$ for any x, y .
- Show there exists a constant $C > 0$ so that $Cd_\infty(x, y) \geq d_1(x, y)$ for all x, y .
- Show that if a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous with respect to any of these metrics, it is continuous with respect to all of them.