Math 25a/55a – Honors Advanced Calculus and Linear Algebra
Solution Set A

Problem 1

We need to prove that d as defined in the problem set is a metric on $S^n(X)$. In order to do that it is enough to check the definition that was given in class. It is clear that $d\colon S^n(X) \times S^n(X) \to \mathbb{R}$ takes on only nonnegative values and also $d(A,A) = |A \setminus A| = 0$. Also we note that $d(A,B) = 0$ implies that $A \setminus B = \emptyset$ or equivalently we have $A \subset B$. Since $|A| = |B| = n$ we deduce that $A = B$.

In order to check symmetry we note that

$$d(A,B) = |A| - |A \cap B| = n - |A \cap B|$$

and the same formula works for $d(B,A)$.

Now the triangle inequality! Let A, B, C be three n-element subsets of X. We need to prove that $d(A,B) \leq d(A,C) + d(C,B)$ or using the definition

$$|A \setminus B| \leq |A \setminus C| + |C \setminus B|$$

First we claim that $A \setminus B \subset (A \setminus C) \cup (C \setminus B)$. Indeed, if $x \in A \setminus B$ (assuming that this set is not empty) we have $x \in A$ and $x \notin B$. If $x \in C$ then $x \in C \setminus B$ and if $x \notin C$ then $x \in A \setminus C$ so in all cases we have $x \in (A \setminus C) \cup (C \setminus B)$ and this proves our claim. Now considering the cardinals of the involved sets we get

$$|A \setminus B| \leq |(A \setminus C) \cup (C \setminus B)| \leq |A \setminus C| + |C \setminus B|$$

which completes the proof.

Problem 2

We have $d'(x,y) := f(d(x,y))$. Then

$$d'(x,x) = f(d(x,x)) = f(0) = 0, \forall x \in X.$$

Also, since

$$d(x,y) > 0, \forall x \neq y \in X,$$

and since $f > 0$ on $(0,\infty)$, we have $d'(x,y) = f(d(x,y)) > 0, \forall x \neq y$ in X. I used that $f(0) = 0 < f(x)$ for all $x > 0$. Symmetry: $d'(x,y) = f(d(x,y)) = f(d(y,x)) = d'(y,x), \forall x, y \in X$. Triangle inequality (note that we use triangle inequality for d):

$$\begin{aligned} d'(x,y) + d'(y,z) &= f(d(x,y)) + f(d(y,z)) \\ &\geq f(d(x,y) + d(y,z)) \\ &\geq f(d(x,z)) = d'(x,z) \end{aligned}$$

We have thus proved that d' is a distance.

Rudin

We see that d_1 is not a distance because the triangle inequality is not verified. Take $x = 1, y = 0, z = -1$ and see that $4 = d_1(x,z) > d_1(x,y) + d_1(z,y) = 2$. Also d_3 is not a metric $d_3(x,y) = 0$ does not imply that $x = y$ (for example we may take $x = 1$ and $y = -1$). This is quite an important condition in the definition of a metric which cannot be left out. Also d_4 is not a metric since $d_4(0,1) \neq d_4(1,0)$. The remaining two are metrics on $\mathbb{R}$. Indeed if we consider the usual metric on $\mathbb{R}$ namely $d(x,y) = |x - y|$ we can deduce that d_2, d_5 are metric using the previous problem.

Take for example $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $f(x) = x^{1/2}$ and $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $g(x) = \frac{x}{x+1}$. We see that $d_2(x,y) = f(d(x,y))$ and $d_5(x,y) = g(d(x,y))$. Also $f(0) = g(0) = 0$ and f is increasing. It is easy to see that g is increasing. So if we prove that $f(x+y) \leq f(x) + f(y)$ and $g(x+y) \leq g(x) + g(y)$ then by applying the result of the previous problem we actually get that d_2 and d_5 are metrics.

Take $x, y \in \mathbb{R}$. I won't check that $f(x+y) \leq f(x) + f(y)$ since this should be trivial for all of you. The fact that $g(x+y) \leq g(x) + g(y)$ follows immediately if we notice that

$$g(x+y) = \frac{x}{x+y+1} + \frac{y}{x+y+1} \leq \frac{x}{x+1} + \frac{y}{y+1} = g(x) + g(y)$$

Problem 3

By the triangle inequality we have $d(x,y) \leq d(x,z) + d(z,y)$, implying $d(x,y) - d(x,z) \leq d(z,y)$. Again by the triangle inequality, we have $d(x,z) \leq d(x,y) + d(y,z)$, giving us $d(x,z) - d(x,y) \leq d(y,z)$. Together this establishes $|d(x,y) - d(x,z)| \leq d(y,z)$. (Note: the most commonly occurring error was to conclude this from only one of the inequalities. The triangle inequality must be used twice).

Consider any $x \in X$ and any $\epsilon > 0$. Then we have $|f(x) - f(y)| = |d(x,x_0) - d(y,x_0)| \leq d(x,y)$ by the above. So if we choose our δ to equal ϵ (or anything less if you like), then $d(x,y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. So continuity is established. A common misunderstanding was to write $d(x,y)$ as $|x - y|$. This is incorrect because only the metric d is defined on X. Absolute value (and subtraction too!) do not have any meaning in X.

Problem 4

First fix $x, y \in \mathbb{R}^n$ given by $x = (x_1, x_2, \dots x_n)$ and $y = (y_1, y_2, \dots y_n)$ where $x_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ for all $1 \leq i \leq n$. We need to show that $d_1(x, y) \leq d_2(x, y) \leq d_\infty(x, y)$. Let us for simplicity denote $|x_i - y_i|$ by a_i for all $1 \leq i \leq n$ where of course a_i are nonnegative real numbers. Then we need to check that

$$\sum_{i=1}^{n} a_i \geq \left(\sum_{i=1}^{n} {a_i}^2 \right)^{1/2} \geq \max_{1 \leq i \leq n} a_i$$

Let's prove the first part. It is clear that

$$\left(\sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} {a_i}^2 + 2 \sum_{1 \leq i < j \leq n} a_i \cdot a_j \geq \sum_{i=1}^{n} {a_i}^2$$

which is equivalent to the first part of our inequality since both quantities involved are positive. The second part is easier! We note that

$$\sum_{i=1}^{n} {a_i}^2 \geq \left(\max_{1 \leq i \leq n} {a_i}^2 \right) = \left(\max_{1 \leq i \leq n} a_i \right)^2$$

and we get the conclusion. We used here the fact that all numbers a_i where $1 \leq i \leq n$ are nonnegative.

Now we note that if we take $C = n$ we have

$$C d_\infty(x, y) = n \cdot \max_{1 \leq i \leq n} |x_i - y_i| \geq \sum_{i=1}^{n} |x_i - y_i| = d_1(x, y)$$

This proves that all three metrics are equivalent (recall the definition given in class)! So a function that is continuous with respect to any of these three metrics is continuous with respect to the others (this was actually mentioned in lecture). Let's prove this! Take d_1, d_2 two equivalent metrics on a metric space X i.e. there exists two constants $m, M > 0$ such that $m d_1(x, y) \leq d_2(x, y) \leq M d_1(x, y)$ and $f \colon X \to Y$ an arbitrary mapping from X to Y.Here Y is a metric space with the metric d. If f continuous with respect to d_1 iff it is continuous with respect to d_2. Assume for example that f is continuous with respect to d_1. We prove the continuity with respect to d_2 of f at a point $a \in X$ arbitrarily chosen. Choose $\epsilon > 0$ arbitrarily. So we know that with respect to d_1, f is continuous at a meaning that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $d_1(x, a) < \delta$ we have $d(f(x), f(a)) < \epsilon$. Now if I choose $\delta' = m\delta$ we have that if $d_2(x, a) < \delta'$ then $d_1(x, y) < \delta$ and thus $d(f(x), f(a)) < \epsilon$ proving the claim. Similarly one can prove that if f is continuous with respect to d_2 then it is continuous with respect to d_1 and we are done!