Math 25a/55a – Honors Advanced Calculus and Linear Algebra
Problem Set B, due Friday September 26.

1. Do Rudin, p. 43 problem 7.

2. Let X be the set whose elements are all sequences $\langle x_n \rangle = (x_1, x_2, \ldots)$ of real numbers. Define $l_1 \subset X$ to be the set of sequences $\langle x_n \rangle$ for which $\sum_1^\infty |x_n|$ is finite, and similarly let l_2 be the set of sequences for which $\sum_1^\infty (x_n)^2$ is finite. Let l_∞ be the space of bounded sequences, i.e. sequences for which there is a number M where $|x_n| \leq M$ for all n.

   (a) Show that $l_1 \subset l_2 \subset l_\infty$. If $\langle x_n \rangle$, $\langle y_n \rangle$ are elements of l_1 and $a \in \mathbb{R}$, show that $\langle x_n + y_n \rangle$ and $\langle a x_n \rangle$ are in l_1. Show the same for l_2 and l_∞.
   Put metrics d_1 on l_1, d_2 on l_2, and d_∞ on l_∞ by the formulas

$$d_1(x, y) = \sum_1^\infty |x_n - y_n|,$$

$$d_2(x, y) = \left(\sum_1^\infty (x_n - y_n)^2 \right)^{1/2}, \text{ and}$$

$$d_\infty(x, y) = \sup_n |x_n - y_n|$$

   (Why are these finite?)

   (b) Let $\langle x^{(i)} \rangle$ be a sequence of points $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$ in l_1. (a sequence of sequences!) Show that if $\lim x^{(i)} = x$ with respect to the metric d_1, then $\lim x^{(i)} = x$ with respect to the metric d_2 also. Similarly show that if $\langle x^{(i)} \rangle$ is a sequence of points in l_2 and $\lim x^{(i)} = x$ with respect to the metric d_2, then $\lim x^{(i)} = x$ with respect to the metric d_∞.

   (c) Consider the following sequence of points of l_1:

$$x^{(1)} = (1, 0, 0, 0, \ldots)$$
$$x^{(2)} = (1/2, 1/2, 0, 0, \ldots)$$
$$x^{(3)} = (1/3, 1/3, 1/3, 0, \ldots)$$
$$\vdots$$

   In general, the nth element $x_n^{(i)}$ of $x^{(i)}$ is $1/i$ if $n \leq i$, and 0 if $n > i$. Show that $x^{(i)}$ converges to the sequence $(0, 0, \ldots)$ in the metrics d_∞ and d_2, but does not converge in the metric d_1. Can you find a sequence of points in l_2 which converge in the metric d_∞ but not in d_2?

   (d) (Hard!) What metrics can you think of to put on the space X of *all* sequences? Can you find one so that a sequence $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots)$ converges to $x = (x_1, x_2, \ldots)$ if and only if for all n we have $x_n^{(i)} \to x_n$ as $i \to \infty$?

3. (a) Let (X, d) be a metric space, $x_0 \in X$, and define

$$f_{\epsilon,x_0}(x) = \begin{cases} 0, & \text{if } d(x, x_0) \geq \epsilon \\ 1 - d(x, x_0)/\epsilon, & \text{if } d(x, x_0) < \epsilon. \end{cases}$$

Show that f_{ϵ,x_0} is continuous.

(b) Suppose that X is not compact, so there is a sequence $\langle a_n \rangle$ in X which has no convergent subsequence. Define

$$f(x) = \sum_{n \geq 1} n f_{1/n,a_n}(x).$$

Show that for any fixed x, there is an $\epsilon > 0$ so that for all but finitely many values of n, the function $f_{1/n,a_n}$ is zero on all points of the ball $B_\epsilon(x)$.

(Because of this, there's no problem making sense of the sum in the definition of f).

(c) Show that f is continuous and unbounded. Thus we see that X is compact if and only if every continuous $f\colon X \to \mathbb{R}$ is bounded.

4. (a) Let D be the set $\{0, 1\}$ with the discrete metric, and let (X, d) be a metric space. Show that there is a continuous surjective map $f\colon X \to D$ if and only if there is a set $Y \subset X$ which is neither X nor the empty set, and which is both closed and open.

If these equivalent conditions hold, we say X is *disconnected*. Otherwise, we call X *connected*.

(b) Show that if $f\colon X \to Y$ is a continuous map between metric spaces and X is connected, then the image set $f(X)$ (with the induced metric from Y) is connected.

(c) Let S be a connected subset of $\mathbb{R}$. Show that S is an interval: either (a, b), $[a, b]$, $(a, b]$, or $[a, b)$, where a is a real number or $-\infty$, and b is a real number or $+\infty$ (of course, if $a = -\infty$ the interval must start with a "(" and if $b = +\infty$ the interval must end with a ")").

(Hint: first show that if $x, y \in S$ and $x < a < y$, then $a \in S$. If S has an upper bound, let $b = \sup S$; otherwise let $b = +\infty$. If S has a lower bound, let $a = \inf S$; otherwise let $a = -\infty$.)

(d) Conversely, show that any interval in $\mathbb{R}$ is connected.

(Hint: let I be an interval, and suppose I is not connected; i.e. suppose $A \subset I$ is both open and closed (as a subset of the metric space I!), and A is neither empty nor all of I. Take $x \in A$, $y \in I \setminus A$, and suppose that $x < y$ (the other case will be the same). Let $z = \sup(A \cap [x, y])$. Show that since A is closed, we must have $z \in A$. Show that this contradicts the fact that A is open.)

(e) Deduce the *intermediate value theorem*: If a continuous function $f\colon [a, b] \to \mathbb{R}$ satisfies $f(a) < 0$, $f(b) > 0$, then there is a $c \in (a, b)$ with $f(c) = 0$.

(f) Show that the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and the interval $[0, 1]$ are not homeomorphic. (You can assume the fact that removing any point from S^1 leaves a connected metric space). If you're feeling energetic, show that $S^1 \backslash \{\text{point}\}$ is homeomorphic to $(0, 1)$! (Use the function $t \mapsto (\cos 2\pi t, \sin 2\pi t)$, $t \in (0, 1)$).