Math 25a–Solution Set 1–October 23, 1997

1. (5 pts.) There were generally very few difficulties with verifying that W is a vector space. The important points are that our 0 vector for W is $(0, 0, 1)$ and the additive inverse $-v$ of an arbitrary vector $v = (x, y, 1)$ is $(-x, -y, 1)$. The second part asking whether W is a subspace brings out some technical issues about subspaces that not all of you seemed to be aware of. W is not a subspace of $\mathbf{R}^3$. To test whether a subset is a subspace, you need to see whether the set is closed under the operations *of the vector space that the subset is contained in.* In this case, W is not closed under either the addition or multiplication of $\mathbf{R}^3$ since $2(x, y, 1) = (2x, 2y, 2)$, which is not in W. Some of you argued that the vector space structure of the subset is different from $\mathbf{R}^3$'s and so cannot be a subspace. While this is good intuition, it is not technically correct. The definition of subspace makes no reference to a vector space structure on the subset. This is really an argument that the vector space structure on the subset would be different from the one inherited from $\mathbf{R}^3$. Can you find a vector space structure on $\mathbf{R}^3$ that agrees with W and makes W a subspace? How about a subspace of a vector space that has a vector space structure that doesn't agree with the one inherited? (Hint for both: shift the zero vector.)

2. (10 pts.) This was very straightforward. I will say only that the zero vector in $\mathcal{L}(V, W)$ is the transformation sending all vectors in V to the zero vector of W and that $-T$ is the transformation sending v to $(-1)T(v)$ for each v in V. Throughout this problem, you are continually using the fact that two transformations are equal iff they take on the same values at each v.

4. (5 pts.) Generally no problems here. You really should, though, note that $W_1 + W_2$ is nonempty. This is true, of course, since W_1 and W_2 are themselves nonempty. Also, you don't need to reverify the vector space axioms to show something is a subspace–you just have to show closure.

5. (5 pts.) Note that $Im(T) = T(V)$ is a subset of W and that $T(v)$ is an element of $T(V)$. Some people made the minor mistake of denoting S as $\{s_1, \ldots, s_n\}$ or as a countable set (indexed by a sequence). But S need not be finite or countable. Linear combinations, however, must be finite. So assume $w = T(v)$. Since v is a linear combination of elts of S, by linearity w is a linear combination of elts of $T(S)$.

6. (10 pts.) $T(a_0 + \cdots + a_3x^3) = a_0 + 2a_1 + a_2 + a_3$. So $p \in ker(T)$ iff $a_0 = -2a_1 - a_2 - a_3$. Notice that to obtain any elt of the kernel, you can choose anything for a_1, a_2, a_3, and then the value of a_0 is forced. With that in mind, we pick $\{x^3 - 1, x^2 - 1, x - 2\}$ as our basis. This is linearly independent by inspection (consider the degrees). It spans for the same reasoning as our intuition above: given an element of the kernel $p = a_0 + \cdots + a_3x^3$, take the linear combination of basis elements with respective coefficients as a_3, a_2, a_1. Check to make sure that the resulting polynomial's constant term agrees with a_0. (The others are obvious.)

7. (5 pts.) Like problem 4, there were virtually no difficulties. Do make sure to note that $G(T)$ is nonempty, which is clear since V is.

3. (10 pts.)

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(a) $\{(1, a, 3), (2, 2, b)\}$ is a linearly dependent set iff $(2, 2, b) = k(1, a, 3)$ for some real number k. Now $(2, 2, b) = k(1, a, 3)$ iff $k = 2$, $a = 1$ and $b = 6$. Thus $\{(1, a, 3), (2, 2, b)\}$ is a linearly

dependent set iff $a = 1$ and $b = 6$.
(b) Let $A = \{(0, 2. -1), (-1, a, b), (2, 0, 3)\}$. It is easy to see that the vectors $(0, 2, -1)$ and $(2, 0, 3)$ are not multiples of each other, hence they are linearly independent. Therefore A is a linearly dependent set iff there exist real numbers α and β such that $(-1, a, b) = \alpha(0, 2, -1) + \beta(2, 0, 3)$. Now

$$(-1, a, b) = \alpha(0, 2, -1) + \beta(2, 0, 3) \iff -1 = 2\beta, a = 2\alpha, b = 3\beta - \alpha \iff$$

$$\beta = -\frac{1}{2}, \alpha = \frac{a}{2}, \alpha = -3\beta - b$$

Hence α and β exist iff $\frac{a}{2} = -\frac{1}{2} - b \iff a + 2b + 3 = 0$. That is, A is linearly dependent if and only if $a + 2b + 3 = 0$.
(c) Let $A = \{(3, a, 1), (1, 0, 1), (b, 0, 0)\}$. The vectors $(3, a, 1)$ and $(1, 0, 1)$ are not multiples of each other, i.e. they are linearly independent. Therefore A is linearly dependent iff there exist real numbers α and β such that

$$(b, 0, 0) = \alpha(3, a, 1) + \beta(1, 0, 1) \iff b = 3\alpha + \beta, a\alpha = 0, \alpha + \beta = 0.$$

In the last set of equations, $\alpha = 0 \Rightarrow \beta = 0 \Rightarrow b = 0$ and it does not impose any constraints on a. Similarly, $a = 0$ does not impose any constraints on b. We can then say in concise form that α and β exist if and only if $ab = 0$, i.e. the set is linearly dependent if and only if $ab = 0$.

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(a) We want to show that $\vec{v_1} = (x_1, x_2), \vec{v_2} = (y_1, y_2)$ span $\mathbf{R}^2$ if and only if $x_1y_2 - y_1x_2 \neq 0$. Assume $x_1y_2 - y_1x_2 = 0$. If $x_2 = 0$, then either $x_1 = 0$ or $y_2 = 0$. Say $x_1 = 0$. $\vec{v_1} = (0, 0)$ and $\vec{v_2} = (y_1, y_2)$ do not span $\mathbf{R}^2$, for $(y_1, y_2 - 1)$ for instance is not in their span. If on the other hand $y_2 = 0$, then $\vec{v_1} = (x_1, 0)$ and $\vec{v_2} = (y_1, 0)$ do not span $\mathbf{R}^2$, as $(0, 1)$ for instance is not in their span. The analysis can be carried out in a similar way if any of the three other (other than x_2) vector components is zero, and we conclude that in all such cases the two vectors don't span $\mathbf{R}^2$.
Assume now that all four components are nonzero. We can write: $\frac{x_1}{y_1} = \frac{x_2}{y_2} \iff (y_1, y_2) = k(x_1, x_2)$ for some real nonvanishing k. The span of $\vec{v_1}$ and $\vec{v_2}$ consists then of all multiples of (x_1, x_2). But $(x_1, x_2 - 1)$ cannot be brought to this form, therefore the span is not all of $\mathbf{R}^2$.
Assume now that $x_1y_2 - x_2y_1 \neq 0$. We can write
$(1, 0) = \frac{y_2}{x_1y_2 - x_2y_1}(x_1, x_2) + \frac{-x_2}{x_1y_2 - x_2y_1}(y_1, y_2)$ and
$(0, 1) = \frac{-y_1}{x_1y_2 - x_2y_1}(x_1, x_2) + \frac{x_1}{x_1y_2 - x_2y_1}(y_1, y_2)$. Since $(1, 0)$ and $(0, 1)$ span $\mathbf{R}^2$, it follows that $\{\vec{v_1}, \vec{v_2}\}$ is a spanning set for $\mathbf{R}^2$ as well.
(b) We will prove the equivalent statement: $x_1y_2 - x_2y_1 = 0$ iff $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent. Suppose first that $x_1y_2 - x_2y_1 = 0$. If $x_2 = 0$ then either $x_1 = 0$ in which case $\vec{v_1} = (0, 0)$ and $\vec{v_2} = (y_1, y_2)$ are clearly dependent, or $y_2 = 0$, in which case there exists a real number k such that $(x_1, 0) = k(y_1, 0)$, which is equivalent to saying that $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent. The cases in which one of x_1, y_1, y_2 is zero, can of course be similarly treated. Assume now that x_1, x_2, y_1, y_2 are all nonzero. We can write $\frac{x_1}{y_1} = \frac{x_2}{y_2} \iff (y_1, y_2) = k(x_1, x_2) \iff \vec{v_1}$ and $\vec{v_2}$ are linearly dependent. Assume now the converse is true: $\vec{v_1}$ and $\vec{v_2}$ are linearly dependent, that is, there is a real k such that

$(y_1, y_2) = k(x_1, x_2)$. $x_1y_2 - x_2y_1 = x_1(kx_2) - x_2(ky_1) = 0$, as desired.
(c) Assume that there is a set of three linearly independent vectors in $\mathbf{R}^2$. Any two of its vectors are then linearly independent and according to part (b), if $\vec{v_1} = (x_1, x_2)$ and $\vec{v_2} = (y_1, y_2)$, then $x_1y_2 - x_2y_1 = 0$. By part (a) now, it follows that $\vec{v_1}$ and $\vec{v_2}$ span $\mathbf{R}^2$, hence $\vec{v_3}$ is expressible as a linear combination of them, meaning that the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent after all.