Math 25a – Honors Advanced Calculus and Linear Algebra
Problem Set 2, due Friday, October 24.

1. CS, p.104 #4
2. CS, p.110 #5
3. CS, p.207 #13, #14
4. CS, p.58 #3
5. CS, p.73 #16-18
6. Suppose V and W are finite dimensional vector spaces, **and** $\dim V = \dim W$. Take linear transformations $T\colon V \to W$ and $S\colon W \to V$, and suppose that $S \circ T = \mathrm{Id}_V$, the identity transformation on V.
   (a) If $\{v_1, \ldots, v_n\}$ is a basis for V, show that $\{Tv_1, \ldots, Tv_n\}$ is a basis for W.
   (b) Show that there exists a linear transformation $S'\colon W \to V$ so that $T \circ S' = \mathrm{Id}_W$.
   (c) Show that $S = S'$. (Hint: compute $S \circ T \circ S'$ in two ways)
   (d) Give examples to show that $T \circ S = \mathrm{Id}_W$ may fail if $\dim V \neq \dim W$.
7. Find a number k and a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^k$ so that $\ker T$ is spanned by the vectors $(2, 0, 1)$ and $(-1, 1, 0)$.
8. In this problem we will give an alternative proof of a key theorem from the lecture.
   (a) Let (a_{ij}) be an $n \times k$ matrix, with $n < k$. Prove that the system of linear equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k & = & 0 \end{array}$$

   has a nonzero solution in $\mathbb{R}^k$.
   (Prove this by induction on n. If $n = 0$, there are no equations. Suppose the result is true for all $n < n_0$. If $n = n_0$, renumber the equations and the variables x_i so that $a_{nk} \neq 0$ – if all the $a_{ij} = 0$ the result is trivial. Then use the last equation to reduce the problem to solving $n - 1$ equations in $k - 1$ unknowns).
   (Note that this can also be proved by using the Rank-Nullity theorem, but we want to use it to prove a result we needed for the RNT).
   (b) Show that if a set $\{v_1, \ldots, v_n\}$ of vectors spans a vector space V, and the set $\{w_1, \ldots, w_k\}$ is linearly independent, then $k \leq n$.
   (Write the vectors w_i as a linear combination of the v_j; if $n < k$, show that the previous result gives a linear dependence among the w's.