Math 25a – Honors Advanced Calculus and Linear Algebra
Problem Set 5, due Monday, November 22.

Problems 1, 2, and 3 are makeup for the exam; hand in solutions directly to me, stapled or paper-clipped to your exam.

1. Suppose $L_1, L_2 \colon V \to \mathbb{R}$ are linear transformations from a vector space V to $\mathbb{R}$, and that the nullspaces (kernels) of L_1 and L_2 are the same. Prove that L_1 is a nonzero multiple of L_2, i.e. there is a nonzero real number a with $L_1 = aL_2$. (You can assume V is finite dimensional, but you are encouraged to try it without this assumption).

2. Suppose a function $f \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{0}$ and that $\vec{v}$ is in the nullspace of $D_{\vec{0}} f$. Define a function $g \colon \mathbb{R} \to \mathbb{R}$ by $g(t) = f(t\vec{v})$. Prove that g is differentiable at 0, and that $g'(0) = 0$.

3. Show that the following four expressions do **not** mean "$f \colon X \to Y$ is not continuous". For each statement you will need to find a function which satisfies the property which is continuous. This will substitute for problem #2, part 2, which was worth three points.

   (a) There exists an $x_0 \in X$ and an $\epsilon > 0$ so that for all $\delta > 0$, $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) \geq \epsilon$.

   (b) There exists an $\epsilon > 0$ so that for all $\delta > 0$ there exist x and x_0 in X so that $d(x, x_0) < \delta$ and $d(f(x), f(x_0)) \geq \epsilon$.

   (c) There exists an $x_0 \in X$ so that for all $\delta > 0$ there exists an $\epsilon > 0$ and an $x \in X$ so that $d(x, x_0) < \delta$ and $d(f(x), f(x_0)) \geq \epsilon$.

   (d) There exists an $x_0 \in X$ and a sequence $\{x_n\}$ converging to x_0 so that the sequence $f(x_n)$ converges to a point y with $y \neq f(x_0)$.

4. CS p. 186 #6 part (a) (Use the Mean Value Theorem)

5. CS p. 92 #6 (look at #5 for the definition of the directional derivative)

6. CS p. 168 #4, 5

7. CS p. 168 #7 (Hint: differentiate with respect to t)

8. (Implicit differentiation) Suppose we have differentiable functions $f \colon \mathbb{R}^n \to \mathbb{R}$ and $F \colon \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying

$$F(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0 \quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

   Calculate the partial derivatives of f in terms of those of F.