Math 25a – Honors Advanced Calculus and Linear Algebra
Solution Set 4, 1997-12-01

Each problem was 10 points. The parts of problem 6 were worth 5 points each.

4 (CS p. 186 #6 part a)) Assume $g'(a) \neq 0$. Given x in some small neighborhood of a, $x \neq a$, we know from the mean value theorem that there is some c between x and a such that $f'(c) = \frac{f(x)-f(a)}{x-a}$, and some d such that $g'(d) = \frac{g(x)-g(a)}{x-a}$. It follows that $f(x)/g(x) = f'(c)/g'(d)$ (where I have used $f(a) = g(a) = 0$). (We are given $g'(a) \neq 0$, so $g'(d) \neq 0$ provided d is close enough to a, since g' is continuous. It follows that $g(x)$ is also nonzero.) As $x \to a$, c and d both $\to a$ since they must lie in (x, a) or (a, x). Since f' and g' are continuous, we find that $f'(c)/g'(d) \to f'(a)/g'(a)$ as $x \to a$, i. e., $f(x)/g(x) \to f'(a)/g'(a)$.

If $g'(a) = 0$ and $f'(a) \neq 0$, then $f(x)/g(x)$ is still $f'(c)/g'(d)$. (My argument that $g(x)$ and $g'(d)$ are nonzero doesn't work anymore, but if there are points x arbitrarily close to a such that $g(x) = 0$, then it's certainly true that the limit we're investigating doesn't exist. If $g(x)$ is nonzero on some neighborhood of a, then $g'(d)$ is also nonzero for those x.) But now the numerator tends to some nonzero value and the denominator tends to zero, so $\lim_{x\to a} f'(c)/g'(d)$ doesn't exist.

6 (CS p. 168 #4, #5)

#4 Defining $g(r, \theta) = (r\cos\theta, r\sin\theta)$, we have $F = f \circ g$. We can apply the chain rule to find the Jacobian of F. We need a notation for the partial derivatives of f; let's let write the function as $f(x, y)$ so we can take the partial derivatives with respect to x and y. When you calculate $J_f J_g$ explicitly you get

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$

and

$$\frac{\partial F}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta,$$

where the partials of F are evaluated at (r, θ) and those of f at $g(r, \theta)$.

#5 This proceeds similarly to the previous problem. We define $g(\rho, \theta, \phi)$ to be $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$. Write f as $f(x, y, z)$ in order to have a notation for the partial derivatives. Apply the chain rule, calculate the various Jacobians explicitly, multiply it out, and you get

$$\frac{\partial F}{\partial \rho} = \frac{\partial f}{\partial x}\sin\phi\cos\theta + \frac{\partial f}{\partial y}\sin\phi\sin\theta + \frac{\partial f}{\partial z}\cos\phi,$$

$$\frac{\partial F}{\partial \theta} = -\frac{\partial f}{\partial x}\rho\sin\phi\sin\theta + \frac{\partial f}{\partial y}\rho\sin\phi\cos\theta,$$

and

$$\frac{\partial F}{\partial \phi} = \frac{\partial f}{\partial x}\rho\cos\phi\cos\theta + \frac{\partial f}{\partial y}\rho\cos\phi\sin\theta - \frac{\partial f}{\partial z}\rho\sin\phi,$$

where the partials of F are evaluated at (ρ, θ, ϕ) and those of f at $g(\rho, \theta, \phi)$.

7 (CS p. 168 #7) Let $g(t) = tv$; we will differentiate $f \circ g$ in two ways. First, using the chain rule, $J_{f\circ g}(t) = J_f(g(t))J_g(t)$. Writing out J_f and J_g explicitly, this is $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tv)$. Note that in taking J_f, we are differentiating the function f, the function that takes (say) w to $f(w)$. But we are differentiating at $g(t)$, i. e., , all the partial derivatives of f are to be evaluated at tv. (In another step, it won't matter whether we're differentiating at tv or v since t will be 1, but in many problems it does matter, and in general you should be careful about it.) We also know that $(f \circ g)(t) = f(tv) = t^k f(v)$; differentiating this function with respect to t, we find $J_{f\circ g}(t) = kt^{k-1}f(v)$. Our two expressions both represent $J_{f\circ g}(t)$ for any t, and in particular they are equal for $t = 1$. So, $kf(v) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(v)$, and we can write v out as $(x_1, \ldots, x_n)$ and get the desired formula.

8 Define $g\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $g(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$. In terms of g, what we are given is that $(F \circ g)(v)$ is zero for all $v \in \mathbb{R}^n$, i. e., $F \circ g$ is the zero function. So its Jacobian is the $1 \times n$ zero matrix, i. e., $0 = J_{F\circ g}(v) = J_F(g(v))J_g(v)$. Writing this out by components, $0 = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial x_{n+1}}\frac{\partial f}{\partial x_i}$ for $i = 1, \ldots, n$. If we impose the additional condition $\frac{\partial F}{\partial x_{n+1}} \neq 0$ (which should have been in the original statement of the problem), then we can rearrange this to get

$$\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i}\bigg/\frac{\partial F}{\partial x_{n+1}}.$$