Math 25a – Honors Advanced Calculus and Linear Algebra
Problem Set 7, due Friday, December 12.

1. CS, p. 180, #7, #8, #11

2. CS, p. 185, #1ade, #2bcd

3. CS, p. 256 #2, #5, #7

4. (a) If W is a subspace of $\mathbb{R}^n$, prove that $\dim W + \dim W^\perp = n$.

   (b) Now let A be an $m \times n$ matrix describing a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Show that the span of the rows of A can be identified with $(\ker T)^\perp \subset \mathbb{R}^n$. Conclude that the "row rank" of A (the dimension of the span of the rows of A) is equal to $\operatorname{rank} T$. (We already knew that the column rank of $A = \operatorname{rank} T$)

5. Suppose $B\colon V \times V \to \mathbb{R}$ is a bilinear function on a normed vector space V. Recall that B is called "positive definite" if for all $\vec{v} \neq \vec{0}$, $B(\vec{v}, \vec{v}) > 0$.

   (a) Show that if B is positive definite then there exists a number $C > 0$ so that $B(\vec{v}, \vec{v}) \geq C \|\vec{v}\|^2$ for all $\vec{v}$; we say that B is "strictly positive definite". (Consider the restriction of $Q(\vec{v}) = B(\vec{v}, \vec{v})$ to the unit sphere; later we will see a purely algebraic proof of this result)

   (b) Now suppose we have a function $f\colon U \to \mathbb{R}$ defined on an open subset $U \subset V$, and that we have a quadratic approximation to f at $\vec{v}_0 \in U$: there is a linear transformation $L\colon V \to \mathbb{R}$ and a bounded bilinear map $B\colon V \times V \to \mathbb{R}$ so that for all $\epsilon > 0$ there exists a $\delta > 0$ so that

   $$\left\|\vec{h}\right\| < \delta \Longrightarrow \left\|f(\vec{v}_0 + \vec{h}) - f(\vec{v}_0) - L(\vec{h}) - B(\vec{h}, \vec{h})\right\| < \epsilon \left\|\vec{h}\right\|^2.$$

   Prove that L is the derivative of f at $\vec{v}_0$.

   (c) Now suppose that $L = 0$ and B is positive definite. Show that f has a local minimum at $\vec{v}_0$.