Math 25a – Honors Advanced Calculus and Linear Algebra
Final Examination, January 23, 1998

Name:

The questions are listed first and then repeated, one to a page. Answer each one on its own page. You can continue an answer on the back of the page and on the blank sheets at the end of the exam. If you need to hand in extra sheets, **write your name on each sheet** that you hand in. Try to make your explanations clear and complete.

This exam is fairly long, so budget your time carefully. All problems are worth the same amount. The later problems are (I think) somewhat harder. Except where otherwise stated, you can use any theorem from the course (be sure to state it carefully!). You can assume an earlier part of a question if it helps you do a later part.

Note: $\mathbb{R}^n$ always carries the standard inner product and its associated norm.

Good Luck!

1. Let X be a metric space, and let $K \subset X$ be a compact subset.

    (a) Give a precise definition of the phrase "K is compact".

    (b) Show directly from the definition that K is bounded.

    (c) Now suppose $X = \mathbb{R}$, and $C \subset \mathbb{R}$ is a closed subset. Show that the set $K + C = \{x + y \mid x \in K, y \in C\}$ is closed.

2. Let the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ -2 & 2 & 3 & 2 \\ 1 & 2 & \frac{3}{2} & 2 \end{pmatrix}.$$

    (a) Find an **orthonormal** basis for each of $\ker T$ and $\operatorname{Im} T$.

    (b) Calculate the 3×3 matrix for the orthogonal projection of $\mathbb{R}^3$ onto $\operatorname{Im} T$.

3. Find, with proof, the points in the unit ball $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ where the function $f(x, y) = x^2 y^3$ takes on its maximum and minimum values.

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

    (a) Show that there exists a unique linear transformation S so that $\langle Tx, y \rangle = \langle x, Sy \rangle$ for all $x, y \in \mathbb{R}^n$.

    (b) Show that if T is invertible, then S is also. (More challenging: show that S and T have the same rank).

5. (a) Carefully state Taylor's theorem for functions $f\colon \mathbb{R} \to \mathbb{R}$.

    (b) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^4$ function, that $f'(0) = f''(0) = f'''(0) = 0$, and $f''''(0) = 1$. Prove that f has a local minimum at 0.

6. Let $f\colon V \to W$ be a function between normed vector spaces.

(a) Suppose that f is differentiable at $\vec{v}_0 \in V$, and the derivative $D_{\vec{v}_0} f$ is a bounded linear transformation (i.e. there is an M so that for all $\vec{v} \in V$, $\|D_{\vec{v}_0} f(\vec{v})\| \leq M \|\vec{v}\|$). Show that f is continuous at $\vec{v}_0$.

(b) Now suppose that f is differentiable at all points of V, and and the derivative is "uniformly bounded", i. e., we can find a single number M which works for all $\vec{v}_0$. Show that f is uniformly continuous.

7. (a) Let $W \subset \mathbb{R}^n$ be a subspace, and take a vector $\vec{v} \in \mathbb{R}^n$ which is not in $W^{\perp}$. Let $\vec{w}_0$ be the unit vector in W which has the largest possible inner product with $\vec{v}$. Show that $\vec{w}_0$ is a multiple of $\text{Pr}_W \vec{v}$, the orthogonal projection of $\vec{v}$ onto W.

(b) Let S be the sphere in $\mathbb{R}^3$ of radius 3 and center $(0,0,0)$. A particle travels with constant speed 1 along a differentiable path $\alpha \colon \mathbb{R} \to S$. At time $t = 0$ it is at the point $(-1, 2, 2)$. If it is moving so as to increase the function $f(x, y, z) = x^2 + xy + yz$ as much as possible, calculate the velocity vector $\alpha'(0)$.

8. Let $T_1, T_2 \colon \mathbb{R}^n \to \mathbb{R}^m$ be two linear transformations, with ranks k_1 and k_2.

(a) Show that $\text{rank}(T_1 + T_2) \leq k_1 + k_2$.

(b) Suppose that $k_1 \leq k_2$. Show that $\text{rank}(T_1 + T_2) \geq k_2 - k_1$.

(c) Give examples where $n = m = 2$, $k_1 = k_2 = 1$, and $\text{rank}(T_1 + T_2)$ is 0, 1, and 2.

9. Let X be a metric space. Recall that we say a sequence of functions $f_n \colon X \to \mathbb{R}$ converges uniformly to a function f if for any $\epsilon > 0$ there exists an $N > 0$ so that if $n \geq N$, then for all $x \in X$ we have $|f_n(x) - f(x)| \leq \epsilon$.

(a) Suppose $\langle f_n \rangle$ is a sequence of differentiable functions defined on $(-1, 1)$, $f_n' \to f'$ uniformly, and $f_n(0) = f(0)$ for all n. Show that $f_n \to f$ uniformly.

(b) Show that if the functions were defined on all of $\mathbb{R}$, we would still get $f_n(x_0) \to f(x_0)$ for all x_0, but the convergence might not be uniform.