Math 25b – Problem Set 2, due Friday, February 13.

1. CS p. 301 #4 (Note: a diffeomorphism is an invertible function such that the function and its inverse are differentiable everywhere)

   (b) (This part was cancelled, but for reference, here's what it *should* have said:) Let $f\colon U \to \mathbb{R}^m$ be an submersion, U open in $\mathbb{R}^n$ (so $n \geq m$). Prove that for each $\vec{v} \in U$ there is an open subset V of U containing $\vec{v}$, an open subset W of $\mathbb{R}^m$ containing $f(\vec{v})$, and a diffeomorphism $h\colon V' \to V$ between an open set $V' \subset \mathbb{R}^n$ and V so that

   $$f(h(x_1, \ldots, x_n)) = (x_1, \ldots, x_m).$$

   (c) Now use part (a) to show that if $f\colon U \to \mathbb{R}^m$ is an immersion and $g\colon U \to \mathbb{R}$ is $\mathcal{C}^1$, then for any $\vec{v} = f(\vec{x}) \in \operatorname{Im} f$ there is an open set V in $\mathbb{R}^m$ containing $\vec{v}$, an open subset V' of U containing $\vec{x}$, and a $\mathcal{C}^1$ function $G\colon V \to \mathbb{R}$ so that $f(V') \subset V$ and for all $\vec{w} \in f^{-1}(V)$ we have $G(f(\vec{w})) = g(\vec{w})$.

   (If we call a function $h\colon \operatorname{Im} f \to \mathbb{R}$ "differentiable" or "$\mathcal{C}^1$" whenever $h \circ f$ is, this shows that we can extend $\mathcal{C}^1$ functions from $\operatorname{Im} f$ to $\mathbb{R}^m$, at least locally).

2. CS p. 211 #6, #7, #8

3. CS p. 214 #2, #3, #4, #5 (look at 11 and 14)

4. CS p. 231 #4

5. (Optional Challenge Problem) Problem 4 on the last set described the curvature of a path $\alpha\colon \mathbb{R} \to \mathbb{R}^n$. Now consider a $\mathcal{C}^2$ function $F\colon \mathbb{R}^2 \to \mathbb{R}$ so that $\operatorname{grad} F \neq 0$ at the point $(0,0)$. We know by the implicit function theorem that the set $F^{-1}(0)$ can be described by a path, at least near $(0,0)$. Try to find, and prove, a formula for the curvature of this curve at $(0,0)$ in terms of the first and second partial derivatives of F there.