Math 25b – Problem Set 3, due Monday, February 23.

1. CS p. 327 #1df, #2, #5.

2. An $n \times n$ matrix is called diagonalizable if it is similar to a matrix (b_{ij}) with $b_{ij} = 0$ whenever $i \neq j$.

   (a) Suppose A and B are 2×2 *diagonalizable* matrices, and that $\det A = \det B$, $\operatorname{tr} A = \operatorname{tr} B$. Show that A and B are similar.

   (b) Find two diagonalizable 3×3 matrices which have the same trace and determinant, but which are not similar.

   (c) Find two 2×2 matrices with the same trace and determinant, but which are not similar (at least one must be non-diagonalizable!).

   (You may find the fact that the identity matrix (or any multiple) is only similar to itself useful in parts (b) and (c))

3. Let A be an $m \times n$ matrix, and suppose we have positive integers $k \leq m, l \leq n$. A $k \times l$ *submatrix* of A is a matrix given by taking k rows and l columns of A, and taking only elements from those rows and columns (in order). For example,

$$\begin{pmatrix} a_{21} & a_{24} & a_{26} \\ a_{41} & a_{44} & a_{46} \\ a_{51} & a_{54} & a_{56} \end{pmatrix}$$

   is a submatrix of A. Determinants of square submatrices of A are called *minors* of A.

   Show that the following statements are all equivalent:

   - $\operatorname{rank} A \geq k$
   - There are k columns of A which are linearly independent.
   - There are k rows of A which are linearly independent.
   - There is a $k \times k$ invertible submatrix of A.
   - A has a nonzero $k \times k$ minor.

   (The first three statement are equivalent by results from last semester. Then show these are equivalent to the fourth)

   Use this to show that the set $\{A \in M(m \times n) \mid \operatorname{rank} A \geq k\}$ is an open subset of $M(m \times n)$ (using any norm on $M(m \times n)$).

4. Given an $n \times n$ matrix $A = (a_{ij})$, let A_{ij} be the matrix obtained by removing the ith row and the jth column of A. Define a matrix $B = (b_{ij})$ by $b_{ij} = (-1)^{i+j} \det A_{ji}$.

   Show that $AB = (\det A) I_n$ (hint: expanding by minors). Conclude that if U is the set of invertible $n \times n$ matrices (which is open by the last problem) and $i \colon U \to U$ is the function given by $i(A) = A^{-1}$, then i is differentiable.