Math 25b – Problem Set 4, due Monday, March 2.

1. CS p. 307 #2, #4, #5 (look at #9)
2. CS p. 317 #4, #6, #7, #8
3. CS p. 323 #1df, #5, #8
4. Let V be an inner product space, and let α be an orthonormal basis. Recall that we call a transformation $T\colon V \to V$ *orthogonal* if $T^*T = TT^* = \text{Id}$.

   (a) Show that the following are equivalent:
   - T is orthogonal.
   - the matrix $A = {}_\alpha T_\alpha$ satisfies $A^t = A^{-1}$.
   - The columns of A form an orthonormal basis of $\mathbb{R}^n$ (with the standard inner product).
   - $\langle \vec{v}, \vec{w} \rangle = \langle T\vec{v}, T\vec{w} \rangle$ for all $\vec{v}, \vec{w} \in V$.
   - T is an *isometry* (distance-preserving): For any $\vec{v} \in V$ $\|T\vec{v}\| = \|\vec{v}\|$, using the norm coming from the inner product on V.

   (You may find the polarization identity useful here – see CS, p.120 #8)

   (b) Show that the Gram-Schmidt orthogonalization process implies that we can write any invertible matrix A as a product OU, where O is an orthogonal matrix and U is upper triangular.

5. In this problem you will show that a transformation $T\colon V \to V$ that has zero as its only eigenvalue is nilpotent. Given a polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$, we can form the transformation $P(T) = a_d T^d + a_{d-1}T^{d-1} + \cdots + a_0 \operatorname{Id}_V$.

   (a) Show that there is a nonzero polynomial P with $P(T) = 0$ (Hint: $\mathcal{L}(V, V)$ is finite dimensional)

   (b) Show that there is a unique nonzero polynomial P_T so that $P_T(T) = 0$, the degree d of P_T is the smallest possible, and $a_d = 1$. We call P_T the *minimal polynomial* of T.

   (c) Suppose 0 is the only eigenvalue of T. Factor P_T into a product of linear terms $(x - \lambda)$ and argue that they must all have $\lambda = 0$, since otherwise $(T - \lambda \cdot \text{Id})$ is invertible. Conclude that $T^d = 0$, where d is the degree of P_T.

   (Notice that by a problem on the review set last semester, we have $d \leq \dim V$, which is much better than what you might expect from part (a)!)