Math 25b — Problem Set 8, due Monday, April 13.

1. CS, p. 436 #7 (remember that our curves are always $\mathcal{C}^1$), and p. 442 #2d, #4d, #5b, #6
2. CS, p. 449 #1bdf, #2, #3
3. Take a $\mathcal{C}^1$ function $f\colon (0,\infty) \to \mathbb{R}$, and define a radially symmetric vector field on $\mathbb{R}^n \setminus \left\{\vec{0}\right\}$ by

$$\vec{F}(\vec{v}) = f(\vec{v})\frac{\vec{v}}{\|\vec{v}\|}.$$

   Show that $\vec{F}$ is path independent by finding a potential function.

4. Let S_n be the n-simplex

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

   (an n-simplex is a generalization of the sequence {point, line segment, triangle, pyramid with triangular base}).

   Calculate the volume of S_n by applying the change of variables

$$\begin{aligned} x_1 &= y_1 y_2 y_3 \dots y_n \\ x_2 &= y_2 y_3 \dots y_n \\ x_3 &= y_3 \dots y_n \\ &\vdots \\ x_n &= y_n \end{aligned}$$

   You will need to check that this defines a $\mathcal{C}^1$ diffeomorphism away from a set of content zero. (There's another way to get this result, which you might want to think about: you can divide the n-cube $[0,1]^n$ into pieces, each of which is the image of S_n under a linear transformation with determinant 1).

5. For any real number a we can define the function $f(x) = x^a$ to be $e^{a \log x}$ for $x > 0$. This agrees with the usual definition when a is an integer, and still satisfies $d/dx(x^a) = ax^{a-1}$.

   (a) For what values of a does $\int_0^\infty x^a dx$ converge? Prove your result. (you will need to handle what happens for x near 0 as well as when $x \to \infty$)

   (b) Define a function g by

$$g(t) = \int_0^\infty x^t e^{-x} dx.$$

   For what values of t is this defined?

(c) Show that $g(t) = tg(t-1)$, and $g(1) = 1$. Thus $g(n) = n!$ if n is a positive integer; g generalizes the factorial function to non-integer values. (Either integrate by parts or differentiate under the integral sign. In either case, you will need to give some argument that you can apply these theorems even though this integral is improper).

(d) Using the change of variables $t = s^2$, show that $g(-1/2) = \sqrt{\pi}$.