(4) As is standard, we denote grad f, div $\vec{n}$ by $\nabla f, \nabla\cdot\vec{n}$ respectively. Beware of the notation dA! As for "$d\theta$," there will in general *not* be a 1-form A whose derivative is dA; if there were, then by Stoke's theorem, the surface area of surfaces without boundary would be zero, which is not the case, as we see below. We drop the absolute value bars in the definition of surface area, since all integrals evaluated are positive.

(a) $\nabla f = (2x, 2y, 2z), \|\nabla f\| = \|(2x, 2y, 2z)\| = 2\|(x, y, z)\| = 2r$, so $\vec{n}(x, y, z) = \nabla f / \|\nabla f\| = \frac{1}{r}(x, y, z)$. Also, $\nabla \cdot \vec{n} = \frac{1}{r}(1 + 1 + 1)\, dx\, dy\, dz = \frac{3}{r}\, dx\, dy\, dz$. Thus, if we denote the ball centered at the origin of radius r by B_r, and note that $S_r = \partial B_r$ we obtain

$$\int_{S_r} dA = \int_{\partial B_r} dA = \int_{B_r} \nabla \cdot dA = \int_{B_r} \frac{3}{r}\, dx\, dy\, dz = \frac{3}{r}\int_{B_r} dx\, dy\, dz = \frac{3}{r} \cdot \frac{4}{3}\pi r^3 = 4\pi r^2$$

(b) Recall that $h^*(\omega \wedge \eta) = (h^*\omega) \wedge (h^*\eta)$ and $h^*dx = d(h^*x)$.

$$\begin{aligned}
h^*x &= r\cos\theta\sin\phi & h^*dx &= r(\cos\theta\cos\phi\, d\phi - \sin\theta\sin\phi\, d\theta)\\
h^*y &= r\sin\theta\sin\phi & h^*dy &= r(\cos\theta\sin\phi\, d\theta + \cos\phi\sin\theta\, d\phi)\\
h^*z &= r\cos\phi & h^*dz &= -r\sin\phi\, d\phi
\end{aligned}$$

We thus obtain, after some algebra, $h^*dA = r^2 \sin\phi\, d\phi\, d\theta$. Thus,

$$\int_{S_r} dA = \int_{[0,2\pi]\times[0,\pi]} h^*dA = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\phi\, d\phi\, d\theta = 2\pi r^2 \int_0^{\pi} \sin\phi\, d\phi = 2\pi r^2(1+1) = 4\pi r^2$$

(c) Showing that $d\omega = 0$ is a straight-forward calculation. Suppose there exists a 1-form η with $d\eta = \omega$. Now, ω restricted to the sphere of radius 1 is dA, and the sphere has no boundary, so

$$4\pi = \int_{S_1} dA = \int_{S_1} \omega = \int_{S_1} d\eta = \int_{\partial S_1} \eta = 0$$

which is a contradiction.