MATH 25A – PROBLEM SET #1
FRIDAY OCTOBER 1

There are three parts in this homework. Please write all three parts separately. They will be graded by different people. When writing down your solutions, don't forget to include complete proofs, not just the answers.

1. Part A

This part of the homework is based on the analysis of curves in $\mathbb{R}^2$ that we did in the first class. Here are all the necessary definitions:

Definition. A *parametrized curve* in $\mathbb{R}^2$ is a continuous map $f : [a, b] \to \mathbb{R}^2$,

$$f(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where *continuous* means that both $x(t)$ and $y(t)$ are continuous functions. (In fact, for most calculations we need these functions to be not only continuous but also differentiable with continuous first derivatives. We will assume this.)

Definition. A parametrized curve as above is *closed* if $f(a) = f(b)$. The curve is *simple* if a and b are the only distinct points in $[a, b]$ mapping to the same point in $\mathbb{R}^2$.

1. We saw in class that the area bounded by a simple closed curve as above, oriented counterclockwise, is given by

$$A = -\int_a^b y(t)x'(t)dt, \qquad \text{or} \qquad B = \int_a^b x(t)y'(t)dt.$$

   Prove that these integrals are equal.
2. Compute the area of the Cardioid $f : [0, 2\pi] \to \mathbb{R}^2$,

$$f(t) = \begin{pmatrix} (1 + \cos t)\cos t \\ (1 + \cos t)\sin t \end{pmatrix}.$$

3. Let $f : [a, b] \to \mathbb{R}^2$ and $g : [c, d] \to \mathbb{R}^2$ be two simple closed curves in $\mathbb{R}^2$. We say that f is a *reparametrization* of g if there exists a continuous function $\psi : [a, b] \to [c, d]$ which maps a to c, b to d, and has a continuous inverse $\psi^{-1} : [c, d] \to [a, b]$, such that

$$f(t) = g(\psi(t)).$$

   (a) Prove that if f is a reparametrization of g then f and g have the same image in $\mathbb{R}^2$.
   (b) Show that if f is a reparametrization of g then the area bounded by f is the same as the area bounded by g. Again, to do calculations with ψ, you may assume that it has a continuous derivative, and the same for its inverse.

2. Part B

1. Using the definition of continuity, prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous at $x = 0$. In other words, for any $\epsilon > 0$ you have to find $\delta > 0$ such that ... . (Hint: First you may want to consider the function $g(x) = x \sin \frac{1}{x}$ if $x \neq 0$, $g(0) = 0$.)

2. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if and only if for any sequence (a_n) converging to a we have that the sequence $(f(a_n))$ converges to $f(a)$. (Hint: here is one way to organize your proof: first show that if f is continuous at a and (a_n) is a sequence converging to a then $(f(a_n))$ converges to $f(a)$. For the converse, show that if f is not continuous then one can construct a sequence (a_n) converging to a such that $(f(a_n))$ does not converge to $f(a)$.)
3. Brouwer Fixed Point Theorem: Exercise 0.5.5 in the textbook.

3. Part C

1. Let V be a vector space.
   (a) Prove that if $\{W_i\}_{i \in I}$ is any collection of subspaces of V then the intersection
   $$\cap_{i \in I} W_i$$
   is again a subspace of V.
   (b) Prove that if W_1 and W_2 are subspaces of V then the union
   $$W_1 \cup W_2$$
   is a subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.
2. Let V be a vector space and $S \subset V$ a subset (not necessarily a subspace). Define $Span(S) \subset V$ to be the intersection of all subspaces of V containing S.
   (a) Prove that $Span(S)$ is a subspace of V.
   (b) Prove that $Span(S)$ is the smallest subspace of V containing S. That means, if W is any subspace of V such that $S \subset W$ then $Span(S) \subset W$.
   (c) Prove that every vector $v \in Span(S)$ can be written as a finite linear combination of vectors in S:
   $$v = a_1 s_1 + \ldots a_n s_n, \qquad a_i \in \mathbb{R}, \quad s_i \in S, \quad n \in \mathbb{N}.$$
   (Hint: show that the set of all such linear combinations is the minimal subspace of V containing S. Why does it follow then that this subspace is equal to $Span(S)$?)
3. Exercise 1.2.22 part 2 in the textbook: What 2×2 matrices A satisfy $A^2 = I$? Your description of these matrices should look something like this: There are three sets of such matrices. A matrix in the first set is determined by two real numbers u and v, where the entries of the matrix in terms of u and v are ... . A matrix in the second set has the form ... .