MATH 25A – PROBLEM SET #2
FRIDAY OCTOBER 8

1. Part A

1. Let V and W be two vector spaces, and let $Hom(V, W)$ be the set of linear transformations from V to W.
   (a) Define addition and scalar multiplication in $Hom(V, W)$ the same way as we did in $C(\mathbb{R})$. Show that these operations are well-defined, namely that the sum of two linear transformations is a linear transformation, and the same for scalar multiplication. Finally, show that $Hom(V, W)$ is a vector space.
   (b) The space $Hom(V, \mathbb{R})$ is called the dual space of V and denoted by V^*. What is $(\mathbb{R}^n)^*$?
2. Linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$.
   (a) Find a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$L \circ L = Id$$

   which is NOT a reflection with respect to a line through the origin or the rotation by 180°. Let S and B be the regions defined by

$$S = \{\begin{bmatrix} x \\ y \end{bmatrix} | 1 \leq x \leq 2, 1 \leq y \leq 2\}, \qquad B = \{\begin{bmatrix} x \\ y \end{bmatrix} | (x+2)^2 + (y-2)^2 = 1\}.$$

   Draw S, B, and their images under L.
   (b) Let A be an invertible 2×2 matrix, and let l be a line in $\mathbb{R}^2$, not necessarily through the origin. If L is the linear transformation defined by A, show that the image $L(l)$ is again a line in $\mathbb{R}^2$.
   (c) Find an example of a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ and a line l in $\mathbb{R}^2$ such that $L(l)$ is not a line.
3. Let $\mathbb{C}$ be the set of complex numbers:

$$\mathbb{C} = \{x + iy | x, y \in \mathbb{R}, i^2 = -1\}.$$

   If you haven't seen complex numbers before, Section 0.6 in the book describes how to add and multiply them. One can see that $\mathbb{C}$ is a vector space (you don't have to prove it, just convince yourself that it is true).
   (a) Show that the map $\phi : \mathbb{C} \to Mat(2, 2)$,

$$\phi(x + iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

   is a linear transformation.
   (b) Show that ψ respects multiplication in $\mathbb{C}$ and $Mat(2, 2)$:

$$\phi(z_1 z_2) = \phi(z_1)\phi(z_2).$$

   (c) Show that ϕ maps inverses to inverses:

$$\phi(\frac{1}{z}) = \phi(z)^{-1}.$$

2. Part B

The goal of the following excercises is to sort out different ways of defining the length of a vector.

Definition. Let V be a vector space. A norm on V is a function $\| \; \| : V \to \mathbb{R}$ satisfying the following properties:

$\|\vec{v}\| \geq 0$ for all $\vec{v} \in V$,
$\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$,
$\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$ for all $\alpha \in \mathbb{R}, \vec{v} \in V$,
$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ for all $\vec{v}, \vec{w} \in V$.

The last inequality is called the triangle inequality. A vector space with a norm is called a normed vector space.

1. Show that the following are normed vector spaces. (Here $C([0,1])$ is the vector space of continuous functions $f : [0,1] \to \mathbb{R}$. You can assume that all such functions have an integral and a maximum value. We will prove these statements later.)
   (a) $\mathbb{R}^n$, $\|\vec{v}\| = |v_1| + |v_2| + \ldots + |v_n|$;
   (b) $\mathbb{R}^n$, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$;
   (you don't need to show the triangle inequality; we will prove it in class);
   (c) $\mathbb{R}^n$, $\|\vec{v}\| = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$;
   (d) $C([0,1])$, $\|f\| = \int_0^1 |f(x)|dx$;
   (e) $C([0,1])$, $\|f\| = \max_x |f(x)|$.

   Definition. Two norms $\| \; \|_1$ and $\| \; \|_2$ on the same vector space V are said to be *equivalent* if there exist constants $a, b > 0$ such that

   $$\|\vec{v}\|_1 \leq a\|\vec{v}\|_2$$

   $$\|\vec{v}\|_2 \leq b\|\vec{v}\|_1$$

   for all $\vec{v} \in V$.

2. Let the norms in 1(a), 1(b) and 1(c) be $\| \; \|_1$, $\| \; \|_2$ and $\| \; \|_\infty$, respectively.
   (a) Show that $\| \; \|_1$ and $\| \; \|_\infty$ are equivalent.
   (b) Show that $\| \; \|_2$ and $\| \; \|_\infty$ are equivalent.
   (c) Does it follow from (a) and (b) that $\| \; \|_1$ and $\| \; \|_2$ are equivalent?
   (d) The norms in 1(d) and 1(e) are not equivalent. Prove this by finding a sequence of functions (f_n) in $C([0,1])$ such that $\int_0^1 |f_n(x)|dx \to 0$ as $n \to 0$, but $\max_x |f_n(x)| = 1$ for all n.

   We define convergence and continuity for maps between normed vector spaces the same way as in case of $\mathbb{R}$, replacing the absolute value $| \; |$ with norms $\| \; \|$. Here is an example:

   Definition. Let $(V, \| \; \|)$ be a normed vector space. A sequence $\vec{v}_1, \vec{v}_2, \ldots$ of vectors in V converges to a vector $\vec{v} \in V$ if for any $\epsilon > 0$ there exists N such that

   $$\|\vec{v} - \vec{v}_n\| < \epsilon$$

   for all $n > N$.

3. (a) Let V be a vector space with two equivalent norms $\| \; \|_1$ and $\| \; \|_2$. Show that a sequence converges according to the first norm if and only if it converges according to the second norm.
   (b) Let $(\mathbb{R}^n, \| \; \|)$ be the normed space in 1(c). Show that a sequence of vectors in $\mathbb{R}^n$ converges with respect to this norm if and only if the n sequences of real numbers formed by the components of the vectors all converge.
   (c) Let $\| \; \|_1$ and $\| \; \|_2$ be the two norms in 1(d) and 1(e). Find a sequence of functions in $C([0,1])$ which converges with respect to the first norm, but not with respect to the second one.