# MATH 25A – PROBLEM SET #3
# FRIDAY OCTOBER 15

## 1. Part A

1. **Correction:** *The problem as stated below is wrong. Replace it with Problem 0.4.6 in the textbook.*
   Write a real number $t \in [0,1]$ in the decimal form
   $$t = 0.t_1t_2\dots \qquad t_i \in \{0,1,2,\dots,9\},$$
   and define a map $f : [0,1] \to \mathbb{R}^2$ by
   $$f(0.t_1t_2\dots) = \begin{pmatrix} 0.t_1t_3t_5\dots \\ 0.t_2t_4t_6\dots \end{pmatrix}.$$
   (a) When we write a real number in decimal form, we have to identify a number that ends with all 9's with another one that ends with all 0's as follows:
   $$0.d_1d_2\dots d_n999\dots = 0.d_1d_2\dots(d_n+1)000\dots,$$
   where $d_n \neq 9$. For example $1.00\dots = 0.99\dots$, so we don't need to use the former.
   Prove that $f$ is well defined. That means, if two decimal representations are identified then their images are also identified.
   (b) Prove that the image of $f$ is the square $[0,1]\times[0,1]$. Is $f$ one-to-one?
   (c) Prove that $f$ is continuous.
   (d) Find a continuous map $g : \mathbb{R} \to \mathbb{R}^n$ that is onto.
2. Let $A$ be a $n \times m$ matrix and $b \in \mathbb{R}^n$. Define $f : \mathbb{R}^m \to \mathbb{R}^n$ by
   $$f(x) = Ax + b.$$
   Prove that $f$ is continuous. (We can think of elements of $\mathbb{R}^n$ as either points or vectors, whichever is more convenient. Here we should consider $x$ and $b$ as vectors, so that addition and matrix multiplication are well defined.)
3. Let $A$ be a $n \times m$ matrix. Define the norm of $A$:
   $$\|A\| = \sup_{\vec{0}\neq\vec{v}\in\mathbb{R}^m} \frac{|A\vec{v}|}{|\vec{v}|}.$$
   (a) Prove that this norm always exists.
   (b) Prove that this defines a norm on the vector space $Mat(n,m)$. (Recall normed vector spaces from the previous homework.)
4. Are the following sets open, closed or neither? Explain why.
   (a) The set of rational numbers $\mathbb{Q}$ in $\mathbb{R}^1$.
   (b) Cantor set in $\mathbb{R}^1$, constructed as follows. Start with the interval $[0,1]$, remove its middle third interval $(1/3, 2/3)$, then remove the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining two pieces, and continue like this. After infinite number of steps, the remaining set is called the Cantor set.
   (d) The set $\{a, a_1, a_2, \dots\}$, where $(a_i)$ is a sequence in $\mathbb{R}^n$ converging to $a$. How about the same set without the limit point $a$?
   (d) The image $f(U)$ of an open set $U \subset \mathbb{R}^n$ by a continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$.

5. (a) Find an infinite nested sequence of open balls $B_{R_1}(a_1) \supset B_{R_2}(a_2) \supset B_{R_3}(a_3) \supset \dots$ in $\mathbb{R}^n$ such that the intersection

$$\bigcap_{i=1}^{\infty} B_{R_i}(a_i)$$

is empty. Does the sequence $(a_i)$ converge?

(b) Can you find a sequence of closed balls $\overline{B}_{R_1}(a_1) \supset \overline{B}_{R_2}(a_2) \supset \overline{B}_{R_3}(a_3) \supset \dots$ such that their intersection is empty? (A closed ball is $\overline{B}_R(a) = \{x : |a - x| \leq R\}$, where $R > 0$.)

## 2. Part B

In this part we consider different ways of defining the distance between two points.

**Definition.** Let $S$ be a nonempty set. A *metric* on $S$ is a function $d : S \times S \to \mathbb{R}$ that assigns to two points $p$ and $q$ in $S$ the distance between them, satisfying the following properties:

$d(p, q) \geq 0$ for all $p, q \in S$,
$d(p, q) = 0$ if and only if $p = q$,
$d(p, q) = d(q, p)$ for all $p, q \in S$,
$d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in S$.

A set $S$ with a metric $d$ is called a *metric space.*

1. Let $(V, \| \ \|)$ be a normed vector space. Show that

$$d(\vec{v}_1, \vec{v}_2) = \|\vec{v}_1 - \vec{v}_2\|$$

defines a metric on $V$.

A metric space is much more general than a normed vector space. For example, the set $S$ does not have to be a vector space.

2. If $(S, d_S)$ is a metric space and $T \subset S$ a nonempty subset, show that there is a metric $d_T$ on $T$ defined by

$$d_T(p, q) = d_S(p, q)$$

for $p, q \in T$.

3. Let $S$ be the set of points on Earth's surface (or a slightly deformed sphere). Define $d(p, q)$ to be the length of the shortest path from point $p$ to point $q$ on the surface. Explain why $d$ is a metric. Such shortest paths are called geodesics.

**Definition.** Two metrics $d_1$ and $d_2$ on the same set $S$ are said to be *equivalent* if for any point $p \in S$ every open $\epsilon$-ball $B^1_\epsilon(p) = \{q \in S | d_1(p, q) < \epsilon\}$ at $p$ with respect to the first metric contains an open $\delta$-ball $B^2_\delta(p)$ with respect to the second metric, and vice versa, every open $\epsilon$-ball $B^2_\epsilon(p)$ with respect to the second metric contains an open $\delta$-ball $B^1_\delta(p)$ with respect to the first metric.

4. Show that equivalent norms give rise to equivalent metrics.
5. Let $S^2$ be the two-sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ in $\mathbb{R}^3$. There are two metrics on $S^2$: the first induced by the standard Euclidean distance in $\mathbb{R}^3$ as in 2, and the second defined by the lengths of shortest paths in $S^2$ as in 3. Prove that these metrics are equivalent.