MATH 25A – PROBLEM SET #5
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1. Part A

1. Problem 1.7.8 in the book.
2. Problem 1.7.10 in the book. You also need to justify why this is the derivative.
3. Problem 1.7.21 in the book. We can think of 2×2 matrices as vectors in $\mathbb{R}^4$. Then the determinant defines a map $\det : \mathbb{R}^4 \to \mathbb{R}$. Find its derivative at I_2.

2. Part B.

1. Let $f, g : \mathbb{R}^m \to \mathbb{R}^2$ be differentiable functions. Show that the map $F : \mathbb{R}^m \to \mathbb{R}$:
$$F(x) = \det \begin{bmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{bmatrix}$$
has partial derivatives
$$D_i F(x) = \det \begin{bmatrix} D_i f_1(x) & g_1(x) \\ D_i f_2(x) & g_2(x) \end{bmatrix} + \det \begin{bmatrix} f_1(x) & D_i g_1(x) \\ f_2(x) & D_i g_2(x) \end{bmatrix}.$$
What is the Jacobian matrix of F?
2. Now let $f, g : \mathbb{R}^m \to \mathbb{R}^3$ be differentiable functions, and let $G : \mathbb{R}^m \to \mathbb{R}^3$ be the cross product:
$$G(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \times \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} f_2(x) & g_2(x) \\ f_3(x) & g_3(x) \end{bmatrix} \\ -\det \begin{bmatrix} f_1(x) & g_1(x) \\ f_3(x) & g_3(x) \end{bmatrix} \\ \det \begin{bmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{bmatrix} \end{bmatrix}$$
Find the Jacobian matrix of G. (Ideally, its columns should look like sums of cross products.)

3. Part C.

1. Problem 1.8.6 in the book. It should read $f : \mathbb{R}^2 \to \mathbb{R}$, and you may ignore the arrows on top of D_i. In part (b) you have to change to polar coordinates. Define the change of coordinates map $g : \mathbb{R}^2 \to \mathbb{R}^2$,
$$g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix}.$$
Now if $\phi : \mathbb{R}^2 \to \mathbb{R}$ is any function in x, y-coordinates, the same function in polar coordinates is the composition $\phi \circ g$:
$$\begin{array}{ccccc} \mathbb{R}^2 & \overset{g}{\to} & \mathbb{R}^2 & \overset{\phi}{\to} & \mathbb{R} \\ \begin{pmatrix} r \\ \theta \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \phi\begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

Partial derivatives of ϕ with respect to r and θ are then the partial derivatives of the composition $\phi \circ g$.

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ has an inverse $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$:
$$f \circ f^{-1} = f^{-1} \circ f = Id.$$
If the derivatives $Df(a)$ and $Df^{-1}(f(a))$ exist, how are they related? Prove your claim.

4. Part D

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(0) = 0$ and
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{xy}{x^2+y^2} \qquad \text{if} \quad \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
   (a) Show that both partial derivatives of f exist at all points.
   (b) Show that f is not continuous.
2. Show that the mean value theorem is not true for maps $f : \mathbb{R}^m \to \mathbb{R}^n$, where $n > 1$. (Hint: find a differentiable map f from a line to $\mathbb{R}^2$ or $\mathbb{R}^3$ such that
$$f(b) - f(a) \neq Df(c)(b-a)$$
for any $c \in [a, b]$. It helps to draw the vectors $Df(c)(b-a)$ in the right hand side of the equation. You don't have to give this map in coordinates, it is enough to describe it in words and to draw a picture.)