

**MATH 25A – EXAM #4**  
**FRIDAY JANUARY 14**

1. Let  $S \subset \mathbb{R}^n$  be a nonempty set. Define the distance function

$$d_S : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$x \mapsto \inf_{s \in S} |x - s|$$

- (a) Prove that  $d_S$  is continuous.  
(b) If  $S$  is compact, show that for any  $x \in \mathbb{R}^n$  there exists a point  $s_x \in S$  such that

$$d_S(x) = |x - s_x|.$$

- (c) The same as in (b) but with  $S$  closed, not necessarily bounded.  
(d) Given two disjoint compact sets  $S, T \subset \mathbb{R}^n$ ,  $S \cap T = \emptyset$ , show that there exist disjoint open sets  $U, V \subset \mathbb{R}^n$ ,  $U \cap V = \emptyset$ , such that  $S \subset U$ ,  $T \subset V$ .  
(e) The same as in (d) but with  $S, T$  closed, not necessarily bounded. (Hint: consider first the case when one of the sets, say  $S$ , is compact.)
2. Here is a method to compactify  $\mathbb{R}^n$  by adding one point  $\infty$  to it. Define a topological space  $(S, \tau)$  as follows:

$$S = \mathbb{R}^n \cup \{\infty\},$$

and the topology  $\tau$  consists of all open sets of  $\mathbb{R}^n$  and sets of the form  $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ , where  $C \subset \mathbb{R}^n$  is compact.

- (a) Prove that  $\tau$  is a topology.  
(b) Prove that the topological space  $(S, \tau)$  is compact.
3. In this problem all vector spaces are assumed to be finite dimensional. Recall that the dual space  $V^*$  of a vector space  $V$  is

$$V^* = \text{Hom}(V, \mathbb{R}).$$

- (a) Let  $v_1, \dots, v_n$  be a basis for  $V$ . Define vectors  $v_1^*, \dots, v_n^*$  in  $V^*$  by specifying their action on the basis vectors  $v_j$ :

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Prove that  $v_1^*, \dots, v_n^*$  form a basis of  $V^*$ . This basis is called the dual basis of  $v_1, \dots, v_n$ .

- (b) Prove that  $(V^*)^* = V$  and  $(v_i^*)^* = v_i$ . (This requires identifying elements in  $V$  with linear functions on  $V^*$ .)  
(c) Given a linear map  $L : V \rightarrow W$ , define its dual  $L^* : W^* \rightarrow V^*$  by composing a linear map on  $W$  with  $L$ :

$$L^*(w^*)(v) = w^*(L(v))$$

for  $w^* \in W^*$ ,  $v \in V$ . Let  $v_1, \dots, v_n$  be a basis of  $V$  and  $w_1, \dots, w_m$  a basis of  $W$ , and let the matrix of  $L$  with respect to these bases be  $A$ . Prove that the matrix of  $L^*$  with respect to the dual bases  $v_1^*, \dots, v_n^*$  and  $w_1^*, \dots, w_m^*$  is the transpose of  $A$ .

- (d) Let  $x_1, \dots, x_N$  be vectors in  $\mathbb{R}^n$  that span it (so,  $N \geq n$ ). Define a linear map  $L : \mathbb{R}^N \rightarrow \mathbb{R}^n$  by sending a standard basis vector  $e_i \in \mathbb{R}^N$  to  $x_i$ , for  $i = 1, \dots, N$ . We can think of the inclusion  $\text{Ker} L \subset \mathbb{R}^N$  as a linear map, denoted by  $K$ :

$$\text{Ker} L \xrightarrow{K} \mathbb{R}^N \xrightarrow{L} \mathbb{R}^n.$$

Dualizing this we get

$$(KerL)^* \xleftarrow{K^*} (\mathbb{R}^N)^* \xleftarrow{L^*} (\mathbb{R}^n)^*.$$

The set of vectors  $y_1 = K^*(e_1^*), \dots, y_N = K^*(e_N^*) \in (KerL)^*$  is called the linear transform (or Gale transform) of  $x_1, \dots, x_N$ .

Let  $x_1, \dots, x_6$  be the following six vectors in  $\mathbb{R}^3$ :

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

These vectors are the vertices of the hexagon obtained by intersecting a cube of side 2 with the hyperplane  $x_1 + x_2 + x_3 = 3$ . Find the linear transform  $y_1, \dots, y_6$ . More precisely, you have to find a basis for  $KerL$  so that  $y_i$  can be written in terms of the dual basis.

- (e) Prove that if  $x_1, \dots, x_N$  lie in a hyperplane that does not pass through the origin, then the linear transform  $y_1, \dots, y_N$  satisfies

$$y_1 + \dots + y_N = 0.$$

(A hyperplane in  $\mathbb{R}^n$  consists of all vectors  $v \in \mathbb{R}^n$  satisfying a linear relation  $a_1 v_1 + \dots + a_n v_n = b$  for some constants  $a_i, b \in \mathbb{R}$ , where not all  $a_i$  are zero. The condition that the hyperplane does not contain 0 means that  $b \neq 0$ . For example, in the previous example, all  $x_1, \dots, x_6$  lie in the hyperplane defined by  $a_i = 1, b = 3$ .)

You can prove this statement directly, generalizing the previous example. But you have to be careful with your notation. To simplify it, you can re-order the vectors  $x_i$  if necessary.

There is also a more conceptual way of proving it. Note that the hyperplane in  $\mathbb{R}^n$  is defined by a linear function, that is, an element in  $(\mathbb{R}^n)^*$ . What is the image of this linear function in  $(\mathbb{R}^N)^*$  ?

4. Fix a large cube  $U \in \mathbb{R}^n$ , called the universe ( $U = \{x \in \mathbb{R}^n : |x_i| \leq B\}$  for some  $B \in \mathbb{N}$ ), and consider pavable sets contained in  $U$ . Prove the following:
- If  $S \subset U$  is pavable then  $U \setminus S$  is also pavable.
  - If both  $S, T \subset U$  are pavable, then  $S \cup T$  is also pavable. (Hint: use Theorem 4.3.1 in the book.)
  - If both  $S, T \subset U$  are pavable, then  $S \cap T$  is also pavable.
  - A collection of sets satisfying the three conditions above is called an algebra of sets. A  $\sigma$ -algebra must, in addition, be closed under countable unions. Show that pavable subsets of  $U$  do not form a  $\sigma$ -algebra. In other words, find a countable collection of pavable sets such that their union is not pavable.
5. The next two problems are equivalent: each follows from the other. Prove whichever you find easier and deduce the other from it.
- Prove that  $k$  vectors  $v_1, \dots, v_k$  in  $\mathbb{R}^n$ ,  $k \leq n$ , are linearly independent if and only if their projection to a coordinate  $k$ -plane  $\mathbb{R}^k$  in  $\mathbb{R}^n$  forms a basis.
  - Let  $A$  be a  $n \times m$  matrix. A  $k$ -dimensional minor of  $A$  is the determinant of a  $k \times k$  submatrix (in other words, choose  $k$  columns,  $k$  rows of  $A$ , and take the determinant). Prove that  $A$  has rank  $k$  if all the  $(k+1)$ -dimensional minors are zero, and some  $k$ -dimensional minor is non-zero. (If  $k$  is equal to  $m$  or  $n$  then the first condition is always satisfied.)
  - Let  $W_k \subset Mat(n, m)$  be the set of all matrices of rank  $\leq k$ . Prove that  $W_k$  is topologically closed.
  - If  $n = m$  prove that if  $W_k$  is closed under matrix multiplication.