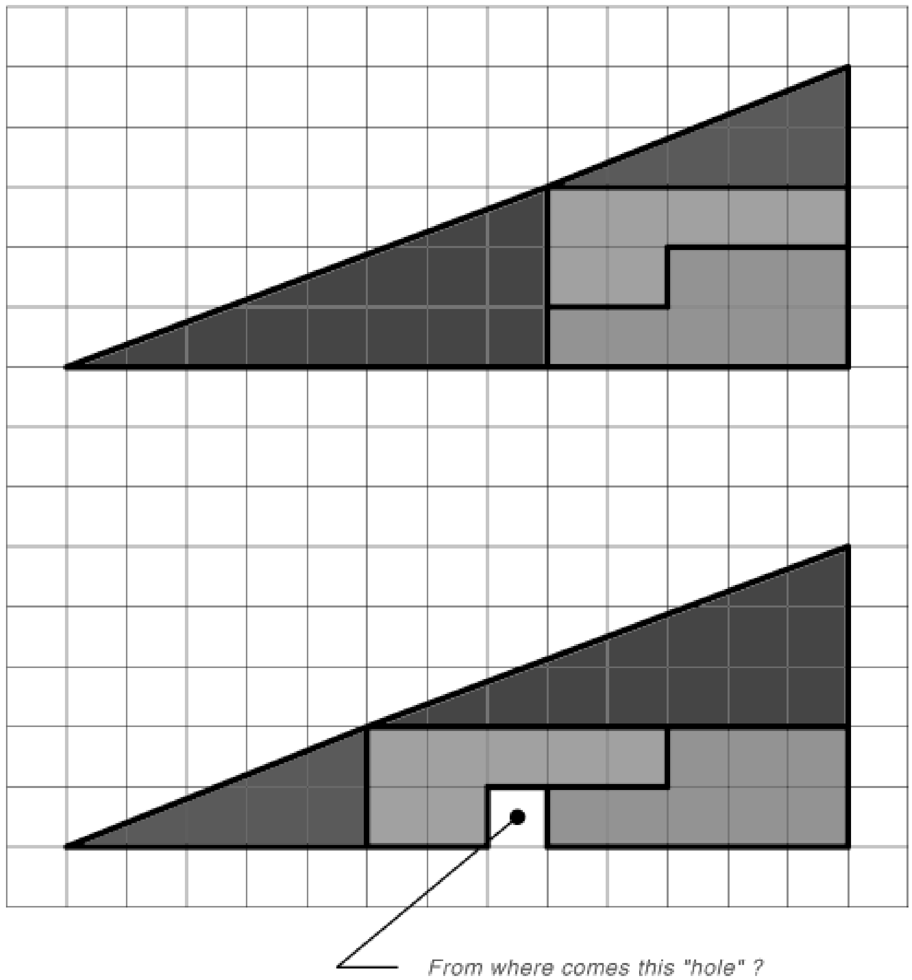
MATH 25A – EXAM #2
FRIDAY NOVEMBER 19

1. Let P_3 be the vector space of polynomials in x of degree at most 3, equipped with the inner product
$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)dx.$$
   (a) Find an orthonormal basis $\{p_1, p_2, p_3, p_4\}$ for P_3.
   (b) The "change of basis" map
$$Id : (V, \{p_1, p_2, p_3, p_4\}) \to (V, \{1, x, x^2, x^3\})$$
   is given by a 4×4 matrix. Find this matrix and its inverse. Write x^3 as a linear combination of p_1, p_2, p_3, p_4.
2. A plane curve of degree d is the set
$$\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,|p(x, y) = 0\},$$
   where $p(x, y)$ is a nonconstant polynomial in x and y of degree d.
   (a) Find a plane curve of degree 2 containing the following 5 points:
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$
   In other words, find a nonconstant polynomial of degree 2
$$p(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$
   such that p vanishes at these 5 points.
   (b) Prove that, given any 5 points in $\mathbb{R}^2$, there exists a plane curve of degree 2 containing these points. Is the same true for any 6 points?
3. Let A be a nilpotent matrix:
$$A^N = 0, \quad A^{N-1} \neq 0,$$
   for some $N > 0$. Prove that $I, A, A^2, \ldots, A^{N-1}$ are linearly independent.
4. Let $A \in Mat(n, n)$.
   (a) Prove that A satisfies a polynomial equation
$$A^d + b_{d-1}A^{d-1} + b_{d-2}A^{d-2} + \ldots + b_1 A + b_0 I = 0,$$
   for some $b_i \in \mathbb{R}$ and $d > 0$.
   (b) A polynomial as above $X^d + b_{d-1}X^{d-1} + \ldots + b_0$ of minimal degree d is called the *minimal polynomial* of A. Find the minimal polynomial of
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
   (c) If $B \in Mat(n, n)$ is invertible, show that BAB^{-1} has the same minimal polynomial as A.
5. Let $A \in Mat(m, n)$, $B \in Mat(n, p)$.
   (a) Prove that
$$Rank(AB) \leq \min(Rank(A), Rank(B)).$$
   (b) If $Rank(B) = n$, prove that
$$Nullity(AB) = Nullity(A) + Nullity(B).$$

HOW CAN THIS BE TRUE ?



Below the four parts are moved around

The partitions are exactly the same, as those used above

From where comes this "hole" ?