FUBINI'S THEOREM

Abstract. My presentation of Fubini's Theorem in class today seemed to rely on the existence of a certain integral. But in fact that integral always exists. Here is a more careful presentation of the argument.

Theorem (Fubini)**.** *Let $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^m$ be rectangles, and $f : A \times B \to \mathbf{R}$ be an integrable function. Suppose that, for each $x \in A$, the function*

$$g_x(y) = f(x, y)$$

is integrable (over B). Then the function

$$A \longrightarrow \mathbf{R}$$
$$x \longmapsto \int_B f(x,y)\,dy$$

is integrable over A, and

$$\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,dy \right) dx.$$

Similarly if, for each $y \in B$, the function

$$h_y(x) = f(x, y)$$

is integrable (over A) then the function

$$B \longrightarrow \mathbf{R}$$
$$y \longmapsto \int_A f(x,y)\,dx$$

is integrable over B, and

$$\int_{A\times B} f = \int_B \left(\int_A f(x,y)\,dx \right) dy.$$

Proof. By symmetry, we just need to prove the first statement. Choose a partition P_A of A, a partition P_B of B and (as in class) form the partition $P = P_A \times P_B$ of $A \times B$. We have

$$\begin{aligned} L(f,P) &= \sum_{s_A \times s_B \in P_A \times P_B} \left(\inf_{(x,y)\in s_A\times s_B} f(x,y) \right) \nu(s_A \times s_B) \\ &\le \sum_{s_A} \sum_{s_B} \inf_{x\in s_A} \inf_{y \in s_B} f(x,y)\,\nu(s_A)\,\nu(s_B) \end{aligned}$$

Date: April 18, 2005.

for the same reasons as in class

$$\leq \sum_{s_A} \inf_{x_A \in s_A} \left(\sum_{s_B} \inf_{y \in s_B} f(x_A, y)\, \nu(s_B) \right) \nu(s_A)$$
$$\leq \sum_{s_A} \inf_{x_A \in s_A} \left(\int_B f(x_A, y)\, dy \right) \nu(s_A).$$

And now exactly the same argument with sups everywhere rather than infs gives

$$\sum_{s_A} \sup_{x_A \in s_A} \left(\int_B f(x_A, y)\, dy \right) \nu(s_A) \leq U(f, P)$$

So

$$L(f, P) \leq \sum_{s_A} \inf_{x_A \in s_A} \left(\int_B f(x_A, y)\, dy \right) \nu(s_A) \leq \sum_{s_A} \sup_{x_A \in s_A} \left(\int_B f(x_A, y)\, dy \right) \nu(s_A) \leq U(f, P),$$

or in other words

$$L(f, P) \leq L\left(x \mapsto \int_B f(x, y)\, dy, P_A \right) \leq U\left(x \mapsto \int_B f(x, y)\, dy, P_A \right) \leq U(f, P),$$

Since f is integrable, we know that

$$\sup_P L(f, P) = \inf_P U(f, P) = \int_{A \times B} f$$

and we can conclude both that

$$x \longmapsto \int_B f(x, y)\, dy$$

is integrable and that

$$\int_{A \times B} f = \int_A \left(\int_B f(x, y)\, dy \right) dx.$$

□

Note that this is essentially identical to the argument that I gave in class: it is just that at the time I did not notice that the argument proved more than I said it did. In fact, this line of reasoning proves still more — we can relax the integrability condition on the "slice functions" $g_x(y)$ and $h_y(x)$ somewhat. See Theorem 3–10 for details.