MATH 25B – PROBLEM SET #1
DUE TUESDAY 15TH FEBRUARY

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Toly) separately from the problems from section 2 (which will be graded by Yan).

1. Toly's problems

(1) *Fields*
  (a) Problem 5 on page 3 of Halmos.
  (b) Problem 7 on page 3 of Halmos.

(2) *Basics*
  (a) Problem 1(c–e) on page 6 of Halmos.
  (b) Problem 3 on page 6 of Halmos.
  (c) Problem 4 on page 12 of Halmos.

(3) *Bases*
  Let S be a subset of a vector space V. Show that the following are equivalent:
  (a) S is a basis for V.
  (b) Every element of V can be written as a linear combination of elements of S in a unique way.
  (c) S is a maximal linearly independent subset of V.
  (d) S is a minimal spanning set in V.
  (We say that a set S is maximal with property P if S has property P and no set T which contains S as a proper subset has property P. Similarly for minimality.)

(4) *Bases: an example*
  The set $M_{2\times 2}(\mathbf{R})$ of 2×2 real matrices over $\mathbf{R}$ forms a vector space over $\mathbf{R}$ (with the usual definitions of addition and scalar multiplication for matrices).
  (a) Show that the subspace W consisting of *symmetric* matrices
  $$W = \{A \in M_{2\times 2}(\mathbf{R}) : A = A^T\}$$
  is a subspace of $M_{2\times 2}(\mathbf{R})$.
  (b) Find a basis for W. What is the dimension of W?
  (c) Find a basis for $M_{2\times 2}(\mathbf{R})$ which contains a basis for W.

(5) $\mathbf{R}$ *as a vector space over* $\mathbf{Q}$.
  Problem 2 on page 16 of Halmos.
  (Problem 2 on page 12 of Halmos may help here.)

2. Yan's problems

(1) *Every vector space has a basis.*
   (a) Find a statement of Zorn's Lemma[1].
   (Zorn's Lemma is perhaps the most useful rephrasing of the Axiom of Choice.)
   (b) Use Zorn's Lemma to show that every vector space contains a maximal linearly independent subset S. From 3(c) above, you know that S is a basis for V.

   This is the "transfinite trickery" mentioned in problem 11 on page 13 of Halmos. You need to use something like Zorn's Lemma here, because the statement "every vector space has a basis" is equivalent to the Axiom of Choice.

(2) *Lagrange interpolating polynomials*
   (a) Write down a polynomial $p \in \mathcal{P}_2$ which vanishes at the point $a_1 \in \mathbf{C}$.
   Write down a polynomial $q \in \mathcal{P}_3$ which vanishes at the points $a_1, a_2 \in \mathbf{C}$.
   (The definition of $\mathcal{P}_n$ is at the top of page 5 of Halmos.
   (b) Let $a_1, \dots a_n \in \mathbf{C}$ be distinct complex numbers. Write down polynomials $f_1, \dots f_n \in \mathcal{P}_n$ such that
   $$f_i(a_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$
   (c) Show that $\{f_1, \dots, f_n\}$ is a basis for $\mathcal{P}_n$.
   Start by showing that they are LI. Then count dimensions.
   (d) Show that given complex numbers $b_1, \dots, b_n$ there is a unique polynomial f of degree at most $n-1$ such that
   $$f(a_i) = b_i \qquad \text{for } i = 1, \dots, n$$

   The polynomials $f_1, \dots, f_n$ are called Lagrange interpolating polynomials.

(3) *Sums of subspaces.*
   (a) Problem 5 on page 19 of Halmos.
   (b) Problem 7 on page 19 of Halmos.
   (c) Problem 8 on page 19 of Halmos.

(4) *Dual spaces*
   (a) Problem 2 on page 22 of Halmos.
   (b) Problem 3 on page 22 of Halmos.
   (c) Problem 5 on page 22 of Halmos.

[1] Google is your friend.