MATH 25B – PROBLEM SET #2
DUE TUESDAY 22ND FEBRUARY

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

1. Yan's problems

(1) *Kernel and range*
Let $T : V \to W$ be a linear map between vector spaces.
  (a) Show that the *kernel* or *null space* of T
$$\ker(T) = \{v \in V : T(v) = 0\}$$
is a subspace of V and that the *range* or *image* of T
$$\mathrm{im}(T) = \{T(v) : v \in V\}$$
is a subspace of W.
  (b) Show that T is injective if and only if $\ker(T) = \{0\}$.
This is a very useful criterion for injectivity.

(2) *Rank–Nullity*
Let V be a finite-dimensional vector space and $T : V \to W$ be a linear map between vector spaces. We define the *rank* of T to be
$$\mathrm{rk}(T) = \dim \mathrm{im}(T)$$
and the *nullity* of T to be
$$\mathrm{null}(T) = \dim \ker(T).$$
Suppose that $V$is finite-dimensional. Pick a basis $\{u_1, \ldots, u_k\}$ for $\ker(T)$ and extend it to a basis $\{u_1, \ldots, u_k, v_1, \ldots, v_l\}$ for V. Show that
$$\{T(v_1), \ldots, T(v_l)\}$$
is a basis for $\mathrm{im}(T)$. Deduce that
$$\mathrm{rk}(T) + \mathrm{null}(T) = \dim(V)$$

(3) *A Corollary*
Suppose that $T : V \to W$ is a linear map between finite-dimensional vector spaces, and that $\dim V = \dim W$. Show that the following are equivalent:
- T is an isomorphism;
- T is injective;
- T is surjective.

So to check if a linear map between finite-dimensional vector spaces of equal dimension is an isomorphism, you just need to check if null(T) *is zero.*

(4) *Exact sequences*
A sequence of linear maps between vector spaces

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \dots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n$$

is called *exact* if and only if

$$\mathrm{im}(T_i) = \ker(T_{i+1}) \qquad \text{for } i = 1, \dots, n-2.$$

Write the zero vector space $\{0\}$ as 0.

(a) Show that

$$0 \longrightarrow V \xrightarrow{T} W \longrightarrow 0$$

is exact if and only if T is an isomorphism.

(b) Under what circumstances are

$$V \xrightarrow{T} W \longrightarrow 0$$

and

$$0 \longrightarrow V \xrightarrow{T} W$$

exact?

(c) Show that if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \dots \longrightarrow V_n \longrightarrow 0$$

is exact then

$$\sum_{i=1}^{i=n} (-1)^i \dim V_i = 0$$

We will return to exact sequences on later homework assignments.

(5) *Dual spaces and solving equations*
(a) Problem 5 on page 27 of Halmos.
(b) Problem 6 on page 27 of Halmos.

(6) *Annihilators*
Problem 8 on page 27 of Halmos.

2. Toly's problems

(1) *Quotient spaces*
   (a) Problem 1 on page 35 of Halmos.
   (b) Problem 4 on page 35 of Halmos.

(2) *First Isomorphism Theorem*
   Explain how and why a linear transformation $T : V \to W$ induces an isomorphism between $V/\ker(T)$ and $\mathrm{im}(T)$.

(3) *Bilinear forms*
   (a) Problem 2 on page 37 of Halmos.
   (b) Problem 4 on page 38 of Halmos.
   (c) Problem 3 on page 38 of Halmos.

(4) *Tensor products*
   (a) Problem 2 on page 41 of Halmos.
   (b) Problem 4 on page 41 of Halmos.