MATH 25B – PROBLEM SET #3
DUE TUESDAY 29TH FEBRUARY

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Toly) separately from the problems from section 2 (which will be graded by Yan).

1. Toly's problems

(1) *Not every tensor can be written as* $v \otimes w$

Let $\{x, y\}$ be a basis for the vector space V. Every element of $V \otimes V$ can be written uniquely in the form

$$a\, x \otimes x + b\, x \otimes y + c\, y \otimes x + d\, y \otimes y \qquad (\star)$$

for some $a, b, c, d \in k$.

(a) Show that $x \otimes y + y \otimes x$ is not equal to $v \otimes w$ for any choice of $v, w \in V$.

(b) For what values of a, b, c, d can the tensor $(\star)$ be written as $v \otimes w$?

(2) *Dual spaces*

Let V be a finite-dimensional vector space. One can define a linear map $T : V \to V^\star$ as follows:

- pick a basis $\{x_1, \ldots, x_n\}$ for V;
- let $\{f^1, \ldots, f^n\}$ be the dual basis for $V^\star$;
- let T be the unique linear map from V to $V^\star$ such that $T(x_i) = f^i$.

(a) Let $V = \mathbf{R}^2$ and take

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Show that $T \begin{pmatrix} a \\ b \end{pmatrix}$ is the element of $V^\star$ given by

$$\begin{array}{ccc} \mathbf{R}^2 & \longrightarrow & \mathbf{R} \\ \begin{pmatrix} c \\ d \end{pmatrix} & \longmapsto & ac + bd \end{array}$$

(b) Let $V = \mathbf{R}^2$ and take

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Compute $T \begin{pmatrix} a \\ b \end{pmatrix}$.

So the map T depends on the choice of basis $\{x_1, \ldots, x_n\}$: it may be "the obvious isomorphism between V and $V^\star$" but it is not canonical. We will show in a later homework assignment that the isomorphism $V \to V^{\star\star}$ is canonical. That homework assignment will also explain precisely what "canonical" means: at the moment you should think of it as meaning "independent of choices".

(3) *Double dual spaces*
Problem 9 on page 28 of Halmos, omitting part (e).

2. Yan's problems

(1) *Extension of scalars*
Problem 5 on page 41 of Halmos.
This is one very common use of tensor products: to "extend the scalars" that we can multiply by. The same idea works for any subfield k of a field k', not just the subfield **R** *of* **C**. *Also, compare with the last problem on the assignment.*

(2) *Halmos's definition of tensor product*
In this question, I will use the same notation as in the handout on tensor products which is posted on the class webpage. Recall that we defined the tensor product

$$V \otimes W = F/Z$$

where F is the free vector space generated by $V \times W$ and Z is the subpace of F defined by

$$Z = span \left\{ \begin{matrix} \delta_{(af_1+bf_2,g)} - a\delta_{(f_1,g)} - b\delta_{(f_2,g)}, \\ \delta_{(f,ag_1+bg_2)} - a\delta_{(f,g_1)} - b\delta_{(f,g_2)} \end{matrix} : \begin{matrix} a, b \in k;\ f, f_1, f_2 \in V; \\ g, g_1, g_2 \in W \end{matrix} \right\}$$

We will identify the dual space of $V \otimes W$, and use this to show that our definition coincides with Halmos' definition when both V and W are finite-dimensional.

(a) Check that one can solve Problem 4(a) on page 35 of Halmos without requiring any of the vector spaces involved to be finite-dimensional.
You do not need to write anything for this bit.

(b) To define a linear map $\xi : F \to k$ we just need to specify the values of ξ on a basis for F. But F has basis

$$\{\delta_{(v,w)} : (v, w) \in V \times W\}$$

so given a map $G : V \oplus W \to k$ we can define a linear map $\xi_G : F \to k$ by setting

$$\xi_G(\delta_{(v,w)}) = G(v, w).$$

Show that if G is bilinear then $\xi \in Z^o$.

(c) Conversely, given $\xi \in F^\star$ we can define a map

$$\begin{aligned} G_\xi : V \oplus W &\longrightarrow k \\ (v, w) &\longmapsto \xi(\delta_{(v,w)}) \end{aligned}$$

Show that if $\xi \in Z^o$ then G_ξ is bilinear.

(d) Combine (b) and (c) to show that Z^o is isomorphic to the vector space $\mathcal{B}$ of bilinear forms[1] on $V \oplus W$. Apply part (a) to give an isomorphism between $(V \otimes W)^\star$ and $\mathcal{B}$.
(e) Deduce that if V and W are finite-dimensional then our definition of $V \otimes W$ coincides with Halmos's definition of $V \otimes W$.

(3) Suppose that V and W are finite-dimensional vector spaces. The set

$$\mathrm{Hom}(V, W) = \{T : V \to W : T \text{ is linear}\}$$

forms a vector space[2]. Construct an isomorphism between $\mathrm{Hom}(V, W)$ and $V^\star \otimes W$.

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[1]This vector space is defined on page 36 of Halmos.
[2]See section 33 of Halmos.