MATH 25B – PROBLEM SET #4
DUE TUESDAY MARCH 8TH

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

1. Yan's problems

(1) *The matrix of a linear transformation*
Let $L \subset \mathbf{R}^2$ be the line $y = mx$. Consider the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which maps the point (x, y) to the point $(p, q) \in L$ which is closest to (x, y). Let β be the standard basis for $\mathbf{R}^2$. Compute the matrix $[T; \beta, \beta]$ of T with respect to the basis β.

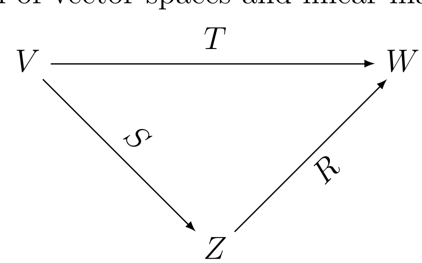
Work in a basis that makes your life easy, and then change basis to get the matrix for T with respect to β. You may assume that T is linear.

(2) *More matrices*
Consider the linear transformations

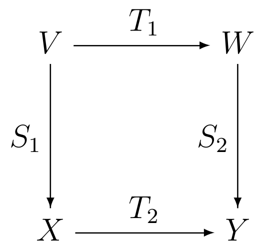
$$S : \mathcal{P}_n \longrightarrow \mathcal{P}_{n+1} \qquad\qquad T : \mathcal{P}_{n+1} \longrightarrow \mathcal{P}_n$$
$$f \longmapsto \int_0^x f(t)\,dt \qquad\qquad g \longmapsto g'$$

(a) Compute the matrix A of S with respect to the standard bases for $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$.
(b) Compute the matrix B of T with respect to the standard bases for $\mathcal{P}_{n+1}$ and $\mathcal{P}_n$.
(c) Compute the matrix products AB and BA.
(d) Why are these not both identity matrices?

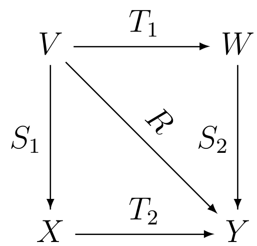
Commutative diagrams
We say that a diagram of vector spaces and linear maps



is *commutative* if and only if $T = RS$. Similarly, we say that such a diagram



is commutative if and only if $S_2T_1 = T_2S_1$, and that



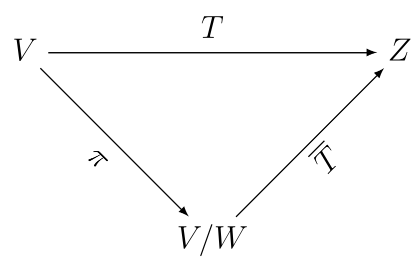
is commutative if and only if $S_2T_1 = R = T_2S_1$. In other words, a diagram of vector spaces and linear maps is commutative if and only if any two compositions of maps which could be equal to each other (*i.e.* which have the same source and target) are equal to each other.

(3) *Quotient transformations*

(a) Let W be a subspace of a vector space V, and let $T : V \to Z$ be a linear map. Let

$$\begin{aligned} \pi : V &\longrightarrow V/W \\ v &\longmapsto v + W \end{aligned}$$

be the natural map from V to V/W. Show that there exists a linear transformation $\overline{T} : V/W \to Z$ such that the diagram



commutes if and only if $W \subset \ker(T)$.

(b) Let W be a subspace of a vector space V and let $T : V \to V$ be a linear map. Show that there exists a linear map $\widetilde{T} : V/W \to V/W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & V \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ V/W & \xrightarrow{\quad \widetilde{T} \quad} & V/W \end{array}$$

commutes if and only if W is invariant under T.

(c) Suppose that W is a subspace of a finite-dimensional vector space V, that $T : V \to V$ is linear, and that W is T-invariant. Pick a basis

$$\gamma_W = \{w_1, \ldots, w_k\}$$

for W, and extend it to a basis

$$\beta = \{w_1, \ldots, w_k, v_1, \ldots, v_l\}$$

for V. We showed in class that the matrix for T with respect to the basis β has the form

$$\left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline 0 & \mathrm{C} \end{array}\right)$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix, 0 is a matrix of zeroes, and C is an $(n-k) \times (n-k)$ matrix.

(i) Define a transformation

$$\begin{aligned} T|_W : W &\longrightarrow W \\ w &\longmapsto T(w) \end{aligned}$$

Show that A is the matrix of $T|_W$ with respect to the basis γ_W.

(ii) Show that C is the matrix of $\widetilde{T} : V/W \to V/W$ with respect to the basis

$$\gamma_{V/W} = \{v_1 + W, \ldots, v_l + W\}$$

for V/W.

(4) *Multilinear forms*

(a) Denote the vector space of multilinear forms from $\overbrace{V \oplus V \oplus \ldots \oplus V}^{r} \to k$ by $\mathrm{Mult_r}$. Show that the symmetrization and skew-symmetrization maps

$$\mathrm{Sym} : \mathrm{Mult_r} \longrightarrow \mathrm{Mult_r} \qquad \mathrm{SkewSym} : \mathrm{Mult_r} \longrightarrow \mathrm{Mult_r}$$

defined in class are respectively projections onto the subspaces of symmetric and skew-symmetric multilinear forms.

(b) Give an example of a vector space V and a non-zero multilinear form in $\ker(\mathrm{Sym})$ for $r = 2$.

(c) Let $V = \mathbf{R}^3$. Write down an alternating trilinear form M on V such that

$$M\left((1,0,0),(0,1,0),(0,0,1)\right) = 5.$$

Can you find more than one such form?

2. Toly's problems

(1) *The dual transformation*

Define $f \in (\mathbf{R}^2)^\star$ by $f(x,y) = 2x + y$ and $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x,y) = (3x + 2y, x)$.

(a) Compute $T^\star(f)$.

(b) Let β be the standard basis for $\mathbf{R}^2$ and $\beta^\star = \{f^1, f^2\}$ be the dual basis. Compute the matrix of $T^\star$ with respect to the basis $\beta^\star$ by finding scalars a, b, c, d such that $T^\star(f^1) = af^1 + bf^2$ and $T^\star(f^2) = cf^1 + df^2$.

(c) Verify that the matrix of $T^\star$ with respect to the basis $\beta^\star$ is the transpose of the matrix of T with respect to the basis β.

(2) *More on duality*

(a) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear map. What are the relationships between the following vector spaces:

$$\ker(T)^o \qquad \ker(T^\star) \qquad \operatorname{im}(T)^o \qquad \operatorname{im}(T^\star)?$$

(b) Let T be a linear map between finite-dimensional vector spaces. Show that

$$\operatorname{rk}(T) = \operatorname{rk}(T^\star)$$

(c) Let

$$0 \longrightarrow V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_n} V_{n+1} \longrightarrow 0$$

be an exact sequence. Show that

$$0 \longrightarrow V_{n+1}^\star \xrightarrow{T_n^\star} V_n^\star \xrightarrow{T_{n-1}^\star} \cdots \xrightarrow{T_1^\star} V_1^\star \longrightarrow 0$$

is also an exact sequence.

(d) *(Optional)* Find someone doing Math 55. Walk past them, deep in conversation with a classmate, saying something like "...well that's just because duality is an exact contravariant functor from the category of vector spaces to itself".

(e) *(Also optional)* Find out what the words in that sentence mean. You could ask Google, or ask me during office hours, or find an introduction to Category Theory in the library, or....

(3) *Inverting matrices using elementary row and column operations*

An elementary row operation on a matrix is one of

(i) Interchanging two rows;
(ii) Adding a multiple of one row to a different row;
(iii) Multiplying a row by a non-zero scalar.

Elementary column operations are the same but with "row" replaced by "column".

(a) Show that if matrices A and B are related by an elementary row operation then there exists an invertible matrix E such that

$$EA = B.$$

Show that if matrices A and B are related by an elementary column operation then there exists an invertible matrix E' such that

$$AE' = B.$$

(b) The *rank* of an $m \times n$ matrix A is the dimension of the subspace of $\mathbf{R}^m$ spanned by the columns of A. Show that the rank of A is equal to the rank of the linear transformation $\mathbf{R}^n \to \mathbf{R}^m$ given by $v \mapsto Av$. Show that the rank of a matrix is left unchanged by elementary row and column operations.
Note that 2(b) above says that the ranks of A and A^t are equal, or that "row rank equals column rank".

(c) Find out what "reduced row-echelon form" of a matrix is. Convince yourself that one can always put a matrix into reduced row-echelon form using a sequence of elementary row operations. Do a couple of small examples.
You do not need to write anything for this bit.

(d) For the remainder of the question assume that A is an invertible $n \times n$ matrix, or in other words an $n \times n$ matrix of rank n. Deduce that one can find a sequence $E_1, \ldots, E_N$ of matrices such that

$$E_N E_{N-1} \ldots E_1 A = I$$

where I is the $n \times n$ identity matrix and the E_i are matrices of the type you constructed in part (a).

(e) Show that if one makes a $2n \times n$ matrix by writing A next to the identity matrix I:

$$B = \begin{pmatrix} A & I \end{pmatrix}$$

and then performs elementary row operations on B until A becomes the identity matrix, then the resulting matrix B' is

$$B' = \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

(f) Compute the inverse of the matrix

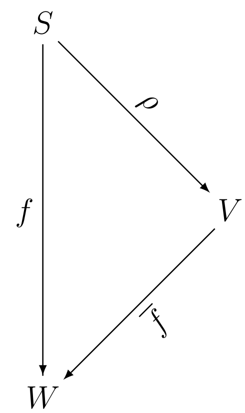
$$\begin{pmatrix} 1 & 6 & 5 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

(g) What happens if you try to use this method to compute the inverse of a non-invertible matrix A?

(4) *Abstract Nonsense*
This exercise gives an introduction to a rather powerful idea: defining things in terms of *universal properties*. We will give a proper definition of the free vector space.

Fix a field k. Define a free vector space over k on a set S to be a pair (V, ρ) such that:

- V is a vector space over k;
- $\rho : S \to V$ is a map (of sets, not of vector spaces: S will in general not be a vector space);
- given any vector space W over k and a map $f : S \to W$, there exists a unique linear map $\overline{f} : V \to W$ such that the diagram



commutes.

(a) First we will show that if a free vector space over k on a set S exists then it is unique up to isomorphism. Suppose that (V, ρ) and (V', ρ') are both free vector spaces over k on S.

(i) Take $W = V$ and $f = \rho$. Show that $\overline{f}$ must be the identity map.

(ii) By taking $W = V'$ and $f = \rho'$, construct a linear map $\overline{\rho'} : V \to V'$. Similarly, construct $\overline{\rho} : V' \to V$. Show that $\overline{\rho'}\,\overline{\rho}$ must be the identity map on V', and that $\overline{\rho}\,\overline{\rho'}$ must be the identity map on V. So the maps $\overline{\rho}$ and $\overline{\rho'}$ are isomorphisms.

Do this by showing that the composite $\overline{\rho}\,\overline{\rho'}$ must play the role of $\overline{f}$ in (a).

Note that this shows that the free vector space over k on S is unique up to isomorphism, and singles out a preferred isomorphism between two free vector spaces over the same field on the same set S. Thus the free vector space on S over k is "unique up to unique isomorphism".

(b) Note that we still do not know that a free vector space on S over k actually exists: we only know that if it exists, it is unique. Let's fix this: given a set S, construct a free vector space on S over k.

(c) Show that every vector space over k is isomorphic to the free vector space on some set S over k.

Arguments with the flavour of part (a) are often called "abstract nonsense".

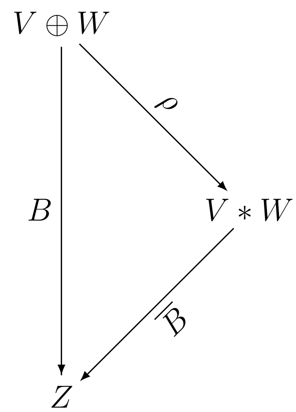
3. An optional problem

(1) *Another definition of tensor products*

This exercise — which is optional, rather difficult, and will not count towards your score on the problem set — gives another, and perhaps the best, definition of the tensor product of two vector spaces. From this point of view, the tensor product is defined by a universal property.

Given vector spaces V and W, we define the TENSOR PRODUCT of V and W to be pair $(V * W, \rho)$ such that

- $V * W$ is a vector space;
- $\rho : V \oplus W \to V * W$ is a bilinear map;
- for any bilinear map $B : V \oplus W \to Z$ there is a unique *linear* map $\overline{B} : V * W \to Z$ such that the diagram



  commutes.
  We could say this as "every bilinear map $V \oplus W \to Z$ factors through $V * W$".

(a) Use abstract nonsense to show that if a TENSOR PRODUCT $(V * W, \rho)$ exists then it is unique up to unique isomorphism.

(b) Show that if we use our definition of tensor product (from the handout), and set

$$\begin{aligned} \rho : V \oplus W &\longrightarrow V \otimes W \\ (v, w) &\longmapsto v \otimes w \end{aligned}$$

then $(V \otimes W, \rho)$ is a TENSOR PRODUCT of V and W.

Now all we need to do is show that Halmos's definition of tensor product is also a TENSOR PRODUCT *and we'll know that it must be isomorphic to ours. We'll do that next time.*

Question 1 in section 2 is from *Linear Algebra* by Friedberg, Insel, and Spence.
Toly suggested the third question in section 2.