MATH 25B – PROBLEM SET #5
DUE TUESDAY MARCH 15TH

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Toly) separately from the problems from section 2 (which will be graded by Yan).

1. Toly's problems

(1) *Examples*

(a) Compute the determinant of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(b) Find the eigenvalues over $\mathbf{R}$ of

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

What are the eigenvalues over $\mathbf{C}$?

(c) Let

$$C = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

Find an invertible matrix Q and a diagonal matrix D such that $C = QDQ^{-1}$.

(2) *The classification of linear transformations from V to W.*

Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces,

(a) Let β be a basis for V, γ be a basis for W, and $A = [T; \beta, \gamma]$. Show that there exist bases β' for V and γ' for W such that $[T; \beta', \gamma'] = B$ if and only if there exist invertible matrices Q and R such that $B = RAQ^{-1}$.
We proved most of this in class already.

(b) Show that there exist bases β for V and γ for W such that $[T; \beta, \gamma]$ takes the form

$$\left(\begin{array}{c|c} \text{I} & \text{Z} \\ \hline \text{Z'} & \text{Z"} \end{array}\right)$$

where I is an $r \times r$ identity matrix and Z, Z', Z'' are respectively $r \times (n-r)$, $(m-r) \times r$, and $(m-r) \times (n-r)$ matrices of zeroes. (Here $\dim V = n$ and $\dim W = m$.)

(c) Suppose that $S : V \to W$ is another linear transformation. Show that there exist isomorphisms $\phi : V \to V$ and $\psi : W \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ V & \xrightarrow{S} & W \end{array}$$

commutes if and only if $\mathrm{rk}(T) = \mathrm{rk}(S)$.

(d) Show that there exist invertible matrices Q and R such that $B = RAQ^{-1}$ if and only if the ranks of the matrices A and B are equal.

This says "linear transformations from V to W are completely classified by their rank".

(3) *Discovering Jordan canonical form*

Suppose that $T : \mathbf{C}^2 \to \mathbf{C}^2$ is a linear transformation such that the only eigenvalue of T is λ. Show that exactly one of the following statements is true:

- there exists a basis β of $\mathbf{C}^2$ such that the matrix of T with respect to β is

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

- there exists a basis β of $\mathbf{C}^2$ such that the matrix of T with respect to β is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

What can you say about a linear transformation $T : \mathbf{C}^3 \to \mathbf{C}^3$ such that the only eigenvalue of T is λ?

(4) *Determinants*

Show that the following two statements are equivalent and then prove them.

(a) If V is a finite-dimensional vector space, $T : V \to V$ is a linear transformation, W is a T-invariant subspace, and $\widetilde{T} : V/W \to V/W$ is the linear transformation induced by T, then

$$\det(T) = \det(T|_W)\det(\widetilde{T}).$$

$\widetilde{T}$ and $T|_W$ were defined in question 3 of the last problem set.

(b) Let A be a $k \times k$ matrix, B be a $k \times n$ matrix, and C be a $(n-k) \times (n-k)$ matrix. Then

$$\det \left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline 0 & \mathrm{C} \end{array} \right) = \det A \det C.$$

Here 0 is a matrix of zeroes.

(5) *Tensor products*
Show that Halmos's definition of the tensor product $V \otimes W$ of finite-dimensional vector spaces makes $V \otimes W$ into a TENSOR PRODUCT of V and W in the sense of the last question on homework 4.
This gives another proof, based on the universal property of tensor product, that Halmos's definition of tensor product and our definition of tensor product coincide up to canonical isomorphism.

2. Yan's problems

(1) *Double-duality is natural*
(a) Let

$$\begin{aligned} \psi_V : V &\longrightarrow V^{\star\star} \\ v &\longmapsto \mathrm{ev}_v \end{aligned}$$

be the map from V to $V^{\star\star}$ constructed in class. Show that for any linear transformation $T : V \to W$ the diagram

$$\begin{CD} V @>T>> W \\ @V{\psi_V}VV @VV{\psi_W}V \\ V^{\star\star} @>T^{\star\star}>> W^{\star\star} \end{CD}$$

commutes.
This is what I meant by "ψ_V is a natural isomorphism[1]": it plays nicely with linear transformations. In categorical language "$V \mapsto \psi_V$ gives a natural transformation between the identity functor **Vect** $\to$ **Vect** *and the double-duality functor* **Vect** $\to$ **Vect***".*
(b) Fix a basis β_V for every finite-dimensional vector space V. Let $\phi_V : V \to V^\star$ be the map constructed as in question 2 of homework 3 but using the basis β_V for V. Exhibit finite-dimensional vector spaces V and W, bases β_V for V and β_W for W, and a linear map $T : V \to W$ such that

$$\begin{CD} V @>T>> W \\ @V{\phi_V}VV @VV{\phi_W}V \\ V^{\star} @<T^{\star}<< W^{\star} \end{CD}$$

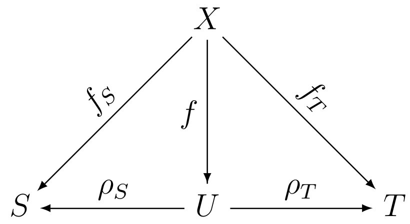
does not commute.
So the isomorphisms ϕ_V are not "natural isomorphisms" in this sense.

[1]NB: ψ_V is only an isomorphism if V is finite-dimensional. But the diagram commutes in the infinite-dimensional case too.

Products and coproducts

We say that (U, ρ_S, ρ_T) is the PRODUCT of finite sets S and T if

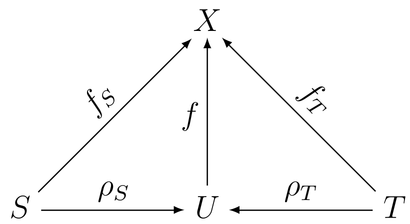
- U is a finite set;
- $\rho_S : U \to S$, $\rho_T : U \to T$ are maps of finite sets;
- if X is a finite set and $f_S : X \to S$, $f_T : X \to T$ are maps of finite sets then there exists a unique map of finite sets $f : X \to U$ such that the diagram



commutes.

We say that (U, ρ_S, ρ_T) is the COPRODUCT of finite sets S and T if

- U is a finite set;
- $\rho_S : S \to U$, $\rho_T : T \to U$ are maps of finite sets;
- if X is a finite set and $f_S : S \to X$, $f_T : T \to X$ are maps of finite sets then there exists a unique map of finite sets $f : U \to X$ such that the diagram



commutes.

In other words, coproduct is just like product except that all the arrows go the other way.

(2) (a) Show that the PRODUCT of two finite sets S and T is unique up to unique isomorphism.

(b) Identify the PRODUCT of two finite sets S and T.
The clue is in the name.

(c) Identify the COPRODUCT of two finite sets S and T.

(d) Identify the PRODUCT of two vector spaces V and W.
The definition of product and coproduct for vector spaces is exactly the same as for finite sets provided that you replace "finite set" by "vector space" and "map of finite sets" by "linear transformation".

(e) Identify the COPRODUCT of two vector spaces V and W.

Your proof for (a) should work for coproducts of finite sets and products/coproducts of vector spaces as well.

(3) *Simultaneous diagonalization*
Suppose that V is a finite-dimensional vector space and that $S : V \to V$ and $T : V \to V$ are diagonalizable linear transformations which commute:

$$ST = TS.$$

Show that S and T are simultaneously diagonalizable, *i.e.* that there exists a basis $\beta = \{v_1, \ldots, v_n\}$ for V such that each v_i is an eigenvector for both S and T.

(4) exp *of a matrix*
Identify the vector space $\mathrm{Mat}_{n\times n}(\mathbf{R})$ with $\mathbf{R}^{n^2}$ using the standard basis. Let $A \in \mathrm{Mat}_{n\times n}(\mathbf{R})$. Define

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

(a) Show that this power series converges in $\mathbf{R}^{n^2}$.
One way to approach this is to bound things in terms of the largest entry in A.
(b) Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

Is A diagonalizable? Compute e^A.
(c) Is it true that

$$e^{B+C} = e^B e^C?$$

The last two questions were suggested by Yan and Toly.